IDENTIFICATION OF INCOMPLETE INFORMATION GAMES IN MONOTONE EQUILIBRIUM

BRENDAN KLINE
UNIVERSITY OF TEXAS AT AUSTIN

Abstract. This paper develops identification results for a class of incomplete information games. These games determine an allocation of units of a valuable object and arrangement of monetary transfers on the basis of the actions taken by the players. The identification strategy is based on the assumption of monotone equilibrium, in which players use strategies that are weakly increasing functions of their valuations for the object being allocated. Such equilibria are known from the economic theory literature to exist under general conditions. The identification result concerns recovering the distribution of valuations for a unit of the object. The identification results flexibly deliver either point identification or partial identification, as appropriate based on the identifying content of the data. The partial identification results are stated as "bounds" on the distribution of valuations in the sense of the usual multivariate stochastic order. The identification results allow for dependent valuations. Moreover, the identification results can apply to an incomplete model that does not necessarily involve a complete specification of all of the details of the game.

JEL codes: C57, D44, D82. Keywords: identification, incomplete information, incomplete model, monotone equilibrium.

1. Introduction

This paper develops identification results for a class of incomplete information games that involve allocation of units of a valuable object and arrangement of monetary transfers on the basis of the actions taken by the players. This class includes models of contests, auctions, procurement auctions and related models of oligopoly competition, strategic (non-"price taking") markets, partnership dissolution, and public good provision. The possible interpretations of the actions include effort in contest models, bids in auction models, bids/asks in market models, or contributions in public good provision models. The actions can be discrete, continuous, or a combination (e.g., a discrete "participation" decision, and if participating, a continuous decision). In some games, as in auctions of a single unit, at most one player can be allocated a unit of the object. In other games, as in auctions of multiple units or public good provision, multiple players can be allocated a unit of the object. In some games, as in contests, the allocation can be non-deterministic.

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Each of the players has a privately-known valuation for a unit of the object, and uses a strategy that relates its valuation to the action it takes in the game. The identification result concerns recovering the distribution of these valuations from the data. The data corresponds to multiple instances ("plays") of the game. The valuations can be dependent, including but not limited to "affiliated values." The partial identification results are stated in terms of "bounds" on the distribution of valuations in the sense of the usual multivariate stochastic order, with point identification as an important special case. The identification results are constructive.

The identification strategy involves using the utility maximization problem to recover information about the unobserved valuation corresponding to an observed action. Hence, the identification strategy relates to an extensive literature in econometrics that uses utility maximization as a source of identification, particularly the use of optimality (e.g., "first order" or equilibrium) conditions in structural models. This approach is especially common in industrial organization, including (but not limited to) in models of firm behavior and monopoly/oligopoly (e.g., Rosse (1970), Bresnahan (1982), Lau (1982), Berry, Levinsohn, and Pakes (1995), Tamer (2003), Aradillas-Lopez (2010), Bajari, Hong, Krainer, and Nekipelov (2010), de Paula and Tang (2012), and Kline (2015, 2016a)) and models of auctions (e.g., Paarsch (1992), Donald and Paarsch (1993, 1996), Laffont, Ossard, and Vuong (1995), Guerre, Perrigne, and Vuong (2000), Athey and Haile (2002), and Aradillas-López, Gandhi, and Quint (2013)). Obviously, these literatures are too large to attempt to fully review here, but have been reviewed in Berry and Tamer (2006), Paarsch and Hong (2006), Athey and Haile (2007), Berry and Reiss (2007), Reiss and Wolak (2007), and de Paula (2013), among other places.

In addition to the differences in the specific class of models under study in this paper compared to in other papers, the identification strategy proposed in this paper differs from existing identification strategies because a main feature is that it is based around the assumption of a monotone strategy equilibrium. This has three important and interrelated consequences for the identification strategy, which are some of the practical contributions of the identification strategy. First, the identification strategy can handle the case of dependent valuations. The assumption of monotone strategy equilibrium is used in particular to deal with players' beliefs in the dependent valuations case. Second, the assumption of monotone strategy equilibrium is broadly applicable, and both intuitive and motivated by a large literature in economic theory. In particular, because the assumption is broadly applicable, the identification strategy can be applied to games that do not point identify the distribution of valuations. Hence, the identification strategy primarily results in partial identification, specifically "bounds" on the distribution of valuations in terms of the usual multivariate stochastic order. Partial identification arises in games in which players use strategies that are *weakly* but not *strictly* monotone. Such strategies involve "flat spots" in which a range of valuations use the same action, so those valuations cannot be distinguished based on observed behavior. In particular, partial identification arises in games with discrete actions (like a discrete "participation" decision). This issue of "flat spots" is exacerbated by the fact that the identification strategy accommodates dependent valuations. Specifically, the beliefs of players with different valuations are generically distinct even if they use the same action, so the identification strategy must account for the fact that players that use the same action do not necessarily have the same beliefs. Sufficient conditions for point identification are provided as an important special case. And third, because the assumption is broadly applicable, the identification strategy applies to a class of incomplete information games studied in this paper, rather than a specific game. These issues are detailed next.

The identification results are based on the assumption of monotone equilibrium. Each player uses a strategy that expresses its action as a function of its valuation. In a monotone equilibrium, the strategies are weakly increasing functions. Therefore, in a monotone equilibrium, if the valuation of a player increases then that player puts forth more effort in contest models, bids more in auction models, offers/demands more in market models, or contributes more in public good provision models. In addition to the intuitive appeal of monotone equilibrium, the economic theory literature has emphasized the importance of proving existence of monotone equilibrium in many specific games. Moreover, the economic theory literature, including Maskin and Riley (2000), Athey (2001), McAdams (2003, 2006), and Reny (2011), has also emphasized the importance of proving general results that establish general conditions on the game that are sufficient for existence of monotone equilibrium. These general results apply to a large class of games. Therefore, the monotone equilibrium assumption can be motivated either as an intuitive assumption, or as a conclusion from the economic theory literature.

The assumption of a monotone strategy equilibrium disciplines the structure of the solution to the utility maximization problem. In particular, with dependent valuations, the utility maximization problem depends on the beliefs held by the player, which depend on the valuation of the player and therefore can be quite complicated, since players with different valuations have different beliefs about the valuations of the other players and hence different beliefs about the actions of the other players. The assumption of a monotone strategy equilibrium makes it possible to recover something about the beliefs of a player, even though the valuation of that player is unobserved by the econometrician. Therefore, the assumption of a monotone strategy equilibrium has a particularly important role in the fact that the identification strategy accommodates dependent valuations. Indeed, this assumption is almost "free" in the special case of independent valuations. The stronger assumption that valuations are independent across players is implausible in many applications. In particular, allowing dependent valuations allows the realistic possibility that a particular player can draw inferences about the valuations of the other players, on the basis of its own private valuation.

This use of monotonicity fundamentally differs from other common uses of monotonicity in econometrics. When assuming monotonicity in other areas of econometrics, monotonicity commonly relates to the functional relationship between two observed variables, and the functional relationship is the object of interest. Monotonicity assumptions are commonly used in regression models or treatment effects models that relate an outcome to a treatment. Monotonicity has been imposed as a shape restriction on the estimator in regression models (e.g., Mukerjee (1988), Ramsay (1988, 1998), and Mammen (1991)), and has been used in the identification of treatment effects models (e.g, Manski (1997), and Manski and Pepper (2000, 2009)). When assuming monotone equilibrium, the monotonicity relates to the equilibrium functional relationship between the observed action and the unobserved valuation, and the distribution of the unobserved valuations is the object of interest. Therefore, in addition to the evident differences in contexts, the role of the monotone equilibrium assumption is fundamentally different from the role of these other common uses of monotonicity assumptions in econometrics.[1]

The identification strategy is based on relatively weak assumptions that can be directly motivated from the economic theory literature, particularly the assumption of a monotone strategy equilibrium. Conversely, the identification strategy avoids making stronger assumptions, like parametric distributional assumptions. Consequently, the identification strategy primarily results in partial identification. Specifically, partial identification results are stated in terms of "bounds" on the distribution of valuations in the sense of the usual multivariate stochastic order. Because the identification strategy accommodates dependent valuations, these bounds involve both the marginal distributions of the valuations of each player and the dependence structure of the valuations across players. The paper also provides stronger sufficient conditions for point identification. Roughly, partial identification arises in games in which players use *weakly* but not *strictly* increasing strategies. And, roughly, point identification arises in games in which players use *strictly* increasing strategies. Intuitively, partial identification is the consequence of players using *weakly* but not *strictly* increasing strategies because that implies that a range of valuations lead to the use of the same action, so those valuations cannot be distinguished based on observed behavior.

Many games do involve the use of *weakly* but not *strictly* increasing strategies. Therefore, focusing only on a point identification strategy would rule out applications to a substantial variety of important games. Focusing instead on a partial identification strategy substantially enlarges the scope for applying the identification result, compared to a point identification strategy.[2]

Specifically, these features of the identification results make it possible to apply the results to games with discrete actions (possibly alongside other continuous actions). Obviously, the strategy necessarily cannot be *strictly* increasing in those games.[3] Many games involve discrete actions, including games

[1]Along these lines of using monotonicity to improve the properties of an estimator, monotonicity of the bidding strategy in specific first-price auction models has been studied in the literature by Henderson, List, Millimet, Parmeter, and Price (2012) and Luo and Wan (2016). Those papers explore the impact of monotonicity on the properties of the estimator (e.g., rate of convergence, optimality, etc.), whereas this paper explores the role of monotonicity in identification. Moreover, those papers work in the context of the important case of specific first-price auction formats, whereas this paper studies partial identification in an entire class of incomplete information games, which flexibly includes auctions that are not necessarily first-price auctions, and also a variety of models other than auctions. Further, those papers assume independent valuations, whereas this paper allows dependent valuations.

[2]Another important identification problem, leading to partial identification, particularly in certain auction formats, concerns the "missing data" problem when the econometrician does not observe the bids of all of the players. Aradillas-López, Gandhi, and Quint (2013) have established partial identification in the important case of an ascending auction with correlated valuations, focusing on showing partial identification of economically relevant seller profit and bidder surplus quantities rather than the object in this paper, the overall joint distribution of valuations. Because the data used by the identification strategy developed here includes the actions of all players, it cannot be applied to address the identification problem studied in Aradillas-López, Gandhi, and Quint (2013). However, the identification strategy developed here does allow "missing data" on other parts of the game, for example the "participation cost" in an auction with a participation cost. Similarly, because the identification strategy can apply to an incomplete specification of the model, the identification results also accommodate "missing *ex ante* knowledge," for example on endogenous quantity functions in an auction. Tang (2011) focuses on partial identification of auction revenue in first-price auctions with common values, which also is not addressed by this paper, which assumes private values.

[3] The "econometrics of entry games" literature with discrete outcomes but (generally) continuous explanatory variables is a different setup in which there is still some continuity in the data to identify possibly infinite-dimensional objects, but in the games studied in this paper, discrete actions implies discrete data. On the other hand, Kline and Tamer (2016) considers the case of partially identified inference with discrete explanatory variables in entry games when the

with a discrete "participation" decision relating to features like reserve prices or participation costs. In that example, the game might include a binary "participation" decision, and if participating, a continuous decision like a bid or ask. Moreover, and more generally, these features of the identification results make it possible to apply the results to games in which there are "flat spots" in the strategies that may arise for various somewhat subtle reasons, even in games without discrete actions. An example of one such case involves a trading game with a single buyer and a single seller. This is a special case of models of strategic (non-"price taking") market behavior, described in Section 2.6. Roughly, if the support of the distribution of valuations for the buyer is different from the support of the distribution of valuations for the seller then the players may use strategies with "flat spots." Other "flat spots" may arise for other reasons in other games, as can be seen from the related economic theory literature. Indeed, the economic theory literature on monotone equilibrium has tended to focus on *weakly* but not necessarily *strictly* increasing strategies.

The issue of the possibility of "flat spots" in the strategies is exacerbated by the fact that the identification strategy accommodates dependent valuations. Specifically, the beliefs of players with different valuations are generically distinct even if they use the same action, so the identification strategy must account for the fact that players that use the same action do not necessarily have the same beliefs.

Also because the identification strategy is based on relatively weak assumptions, the identification result applies to a class of incomplete information games. This class is sufficiently general to include a variety of important specific games. One obvious implication is that this means that it is not necessary to establish identification results for each specific game within the class of incomplete information games studied in this paper. One slightly subtle implication is that this means that the econometrician does not necessarily need to specify a complete model for the game, because the identification strategy applies to an entire class of incomplete information games. In particular, the identification strategy does not depend on the econometrician knowing the details of how the allocations and transfers are determined on the basis of the actions of the players, because it is possible to use the data to identify these objects. The equilibrium strategies that relate a player's valuation to that player's action will implicitly depend on these details. Therefore, if the identification strategy depended on the specific game or the specific details of the game, different identification strategies would potentially be needed for each specification of how the allocations and transfers are determined. For example, the econometrician does not need to know the "contest success function" in models of contests, which relate the effort put forth by the players to the probabilities that each of them win the contest (see Example 1). And for another example, the econometrician does not need to know the endogenous quantity function in auctions where the quantity of the object allocated depends on the actions of the players, as in a "supply curve" (see Example 2).[4] Such features of

parameter of interest is finite-dimensional, rather than the infinite-dimensional distribution of valuations in the games studied in this paper.

[4]Along similar lines of not knowing the details of an auction model, Haile and Tamer (2003) studied the (partial) identification of bidder valuations in an incomplete model of English auctions with symmetric independent private values. See also Chesher and Rosen (2015) for further identification results in a related model of English auctions with symmetric independent private values, based on generalized instrumental variables. Haile and Tamer (2003) studied identification of bidder valuations based on the assumptions that bidders will not be "outbid" and will not "overbid."

the game can be identified from the data, rather than assumed known, and the same identification strategy for the distribution of valuations applies regardless of the details of these features of the game. Moreover, the fact that these features of the game can be identified can be useful by itself, depending on the application, either as an object of interest by itself or as an "ingredient" in an analysis alongside the distribution of valuations.

The remainder of the paper is organized as follows. Section 2 sets up the incomplete information game framework studied in this paper and provides some baseline analysis. Section 3 provides the partial identification strategy. Section 4 provides sufficient conditions for point identification. Finally, Section 5 concludes.

The appendices collect a variety of technical details and extensions. Appendix A discusses sufficient conditions for some of the assumptions. Appendix B provides further examples of the incomplete information games framework studied in this paper. Appendix C discusses the role of equilibrium assumptions in the identification results. This discussion shows that identification of some features of the distribution of valuations is robust to partial failures of the equilibrium assumption. Appendix D provides an extension of the identification strategy under an additional assumption, that is especially useful with many discrete actions, or an entirely discrete action space. Proofs are provided in Appendix E.

2. Allocation-transfer game framework

There are $N \geq 2$ players[5] in the game, which determines the allocation of units of a valuable object and arrangement of monetary transfers on the basis of the actions of the players. Examples are discussed after setting up the framework. Players are indexed by $i = 1, 2, \ldots, N$.

2.1. Utility functions.

2.1. Utility functions. Player i has valuation θ_i for a unit of the object. The utility of player i with valuation θ_i, and who receives allocation x_i of the object and transfers away ("pays") t_i units of money is

$$U(\theta_i, x_i, t_i) \equiv \theta_i x_i - t_i.$$

The sign of t_i is unrestricted, so player i can be "paid" if t_i is negative. For example, the monetary transfer could be the payment in an auction model, the "price" in a market model, or the contribution in a public good provision model. This utility function is standard in the economic theory literature.

It is common knowledge amongst the players that the valuations $\theta \equiv (\theta_1, \theta_2, \ldots, \theta_N)$ are drawn from the joint distribution $F(\theta)$. The actual realization θ_i is the private information of player i.

Assumption 1 (Dependent valuations)**.** *It is common knowledge amongst the players that θ is drawn from $F(\theta)$, and θ_i is the private information of player i. The distribution $F(\cdot)$ has associated ordinary density $f(\cdot)$. For each $i \in \{1, 2, \ldots, N\}$, the support of the distribution of θ_i is convex.*

By contrast, this paper studies identification under the assumption of monotone equilibrium. Consequently, Haile and Tamer (2003) and this paper are two non-nested approaches to different identification problems that share the feature of not requiring the econometrician to specify a complete model. Moreover, the results in this paper considers identification in settings not restricted to English auction formats and settings not restricted to symmetric independent valuations.

[5]In principle, the results could apply to some "single-agent games" with $N = 1$, of course as long as the assumptions hold in such a game, but the focus is on multiple-agent games.

The part of this assumption about the support states the standard condition that the support of θ_i is an interval. The econometrician need not know the support. The identification results allow dependent valuations, which is important because the assumption that valuations are independent across players is implausible in many applications. In particular, allowing dependent valuations allows the realistic possibility that a particular player can draw inferences about the valuations of the other players, on the basis of its own private valuation. The assumption of monotone equilibrium is particularly important in the fact that the identification strategy can accommodate dependent valuations. Nevertheless, the identification results do simplify under the further assumption of independent valuations:

Assumption 2 (Independent valuations)**.** *In addition to Assumption 1, player valuations are independent, in the sense that the components of $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$ are independent random variables, so $F(\theta) = F_1(\theta_1)F_2(\theta_2)\cdots F_N(\theta_N)$.*

Even under the assumption of independent valuations, it is not assumed that players necessarily draw their valuation from the same distribution, so $F_i(\cdot)$ need not equal $F_j(\cdot)$, which is useful for example to model "weak" and "strong" bidders in auctions or asymmetries between buyers and sellers in models of market behavior. Symmetry is allowed as a special case.

2.2. **Actions.** After realizing θ_i, player i takes an action a_i from its action space $\mathcal{A}_i$. The interpretation of the actions depends on the game, and includes efforts in contest models, bids in auction models, announcements (bids/asks) in market models, and contributions in public good provision models.

Assumption 3 (Action space)**.** *For each $i \in \{1, 2, \ldots, N\}$, the econometrician knows the action space for player i is $\mathcal{A}_i \subseteq \mathbb{R}$. Further, $\mathcal{A}_i = \mathcal{A}_{i,disc}^{low} \cup \mathcal{A}_{i,cont} \cup \mathcal{A}_{i,disc}^{high}$ where*

$$\mathcal{A}_{i,disc}^{low} = \{a_i^{(low,1)}, a_i^{(low,2)}, \ldots, a_i^{(low,|\mathcal{A}_{i,disc}^{low}|)}\} \text{ if } \mathcal{A}_{i,disc}^{low} \neq \emptyset$$

and

$$\mathcal{A}_{i,cont} = \begin{cases} [\alpha_i, \beta_i] & \text{if } \alpha_i < \beta_i \text{ are finite} \\ (-\infty, \beta_i] & \text{if } \alpha_i = -\infty \text{ and } \beta_i \text{ is finite} \\ [\alpha_i, \infty) & \text{if } \alpha_i \text{ is finite and } \beta_i = \infty \\ (-\infty, \infty) & \text{if } \alpha_i = -\infty \text{ and } \beta_i = \infty \\ \emptyset & \text{if } \alpha_i > \beta_i \end{cases}$$

and

$$\mathcal{A}_{i,disc}^{high} = \{a_i^{(high,1)}, a_i^{(high,2)}, \ldots, a_i^{(high,|\mathcal{A}_{i,disc}^{high}|)}\} \text{ if } \mathcal{A}_{i,disc}^{high} \neq \emptyset.$$

And for any $a_i^{(1)} \in \mathcal{A}_{i,disc}^{low}$, $a_i^{(2)} \in \mathcal{A}_{i,cont}$ and $a_i^{(3)} \in \mathcal{A}_{i,disc}^{high}$, it holds that $a_i^{(1)} < a_i^{(2)} < a_i^{(3)}$. If any of $\mathcal{A}_{i,disc}^{low}$, $\mathcal{A}_{i,cont}$, and $\mathcal{A}_{i,disc}^{high}$ are empty, this is understood to hold restricted to the non-empty sets.

The action space $\mathcal{A}_i$ includes[6] both a "continuous part" $\mathcal{A}_{i,\text{cont}}$ and a "discrete part" $\mathcal{A}_{i,\text{disc}} \equiv \mathcal{A}_{i,\text{disc}}^{\text{low}} \cup \mathcal{A}_{i,\text{disc}}^{\text{high}}$. Any of $\mathcal{A}_{i,\text{disc}}^{\text{low}}$, $\mathcal{A}_{i,\text{cont}}$, and $\mathcal{A}_{i,\text{disc}}^{\text{high}}$ can be empty sets. The "continuous part" $\mathcal{A}_{i,\text{cont}}$ must

[6]The case $\alpha_i = \beta_i$ is not allowed, in order to guarantee that $\mathcal{A}_{i,\text{cont}}$ is not a finite set.

be non-empty for the main identification strategy to result in non-trivial bounds on the valuations, but the "discrete part" $\mathcal{A}_{i,\text{disc}}$ can be an empty set. Appendix D develops an extension of the identification strategy that is useful in games with many discrete actions, or entirely discrete action spaces.

This incomplete information game framework does not necessarily require a "numerical interpretation" of the actions in $\mathcal{A}_{i,\text{disc}}$, similar to how the numerical encodings of the categories in categorical choice models may or may not actually have a substantive "numerical interpretation." For example, in games with voluntary participation including auctions with participation costs, one of the actions is the "do not participate" action. Hence, in such auctions, $\mathcal{A}_i = \{DNP\} \cup [r_i, \infty)$ so $\mathcal{A}_{i,\text{disc}}^{\text{low}} = \{DNP\}$ and $\mathcal{A}_{i,\text{cont}} = [r_i, \infty)$ and $\mathcal{A}_{i,\text{disc}}^{\text{high}} = \emptyset$, where $r_i \geq 0$ is a lowest allowed bid ("reserve price"). The action "DNP" in such games would have a special ("non-numerical") interpretation of "do not participate (in the auction)." Actions taken in the set $[r_i, \infty)$ would have the usual interpretation as the associated numerical bid. In another example, if the game involves both buyers and sellers, as in the example of strategic market behavior in Section 2.6, then $\mathcal{A}_{i,\text{disc}}^{\text{low}} = \{DNP\}$ for i being a buyer and $\mathcal{A}_{i,\text{disc}}^{\text{high}} = \{DNP\}$ for i being a seller, since buyers with a low valuation would not participate and sellers with a high valuation would not participate.[7]

Even if there is no "numerical interpretation" of the actions in $\mathcal{A}_{i,\text{disc}}$, it is important that the action space is ordered because it is assumed that players use monotone strategies, which requires by definition that the action space must be ordered. The numerical encoding of "special" actions as numbers in $\mathcal{A}_{i,\text{disc}}$ respects the ordering of the actions.[8] In particular, actions in $\mathcal{A}_{i,\text{disc}}^{\text{low}}$ are "lower" than actions in $\mathcal{A}_{i,\text{cont}}$, and actions in $\mathcal{A}_{i,\text{disc}}^{\text{high}}$ are "higher" than actions in $\mathcal{A}_{i,\text{cont}}$. For example, in auctions with voluntary participation, generically players with low valuations choose to not participate, so it makes sense to define DNP to be in $\mathcal{A}_{i,\text{disc}}^{\text{low}}$, so that DNP is a lower action compared to any participating bid in $[r_i, \infty)$, in order for the equilibrium strategy to be monotone.[9]

The vector of all players' actions is $a = (a_1, a_2, \ldots, a_N)$, the vector of all players' allocations is $x = (x_1, x_2, \ldots, x_N)$, and the vector of all players' transfers is $t = (t_1, t_2, \ldots, t_N)$.

2.3. **Allocations and transfers.** The game determines the allocations and transfers on the basis of the actions. Even for a given profile of actions, non-deterministic allocations and monetary transfers are allowed, for example to allow "noise" in the process of determining a winner in a contest (see Example 1).

Let $\mathcal{X} \subseteq \mathbb{R}^N$ be the set of feasible allocations of the units of the object across the N players, or equivalently, the feasible set of values for x. For example, if the game involves the allocation of a single indivisible unit of an object, $\mathcal{X} = \{(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, 0, \ldots, 1), (0, 0, 0, \ldots, 0)\}$, where the last feasible allocation reflects the possibility that the game does not give the object to a player, for example if a reserve price is not met in an auction. Depending on the set $\mathcal{X}$, the framework allows that multiple players are allocated units of the object, for example in the case of

[7]See for example Athey (2001, Section 4.1) or Reny and Zamir (2004) or Menezes and Monteiro (2005, Section 3.1.4) or Tan and Yilankaya (2006) or Araujo and De Castro (2009) among other examples from the economic theory literature on such an action space.

[8]Any two finite totally ordered sets of equal cardinality are order isomorphic, so in particular any finite totally ordered set is order isomorphic to any subset of $\mathbb{Z}$ with equal cardinality and the usual total order on $\mathbb{Z}$.

[9]It could be that DNP is encoded as -1 or -2, for example. The specific numerical encoding is irrelevant.

public good provision or auctions with multiple units. The framework also allows that some players have fractional allocation, for example in the case of divisible objects. Similarly, let $\mathcal{T} \subseteq \mathbb{R}^N$ be the set of feasible transfers across the N players, or equivalently, the feasible set of values for t. Finally, let $\mathcal{O} \subseteq \mathcal{X} \times \mathcal{T}$ be the set of *jointly* feasible allocations and transfers across the N players, or equivalently, the *jointly* feasible set of values for x and t. The combination of x and t is the *outcome of the game*. The econometrician does not need to know $\mathcal{X}$, $\mathcal{T}$, and/or $\mathcal{O}$. Let $\Delta(S)$ be the set of all random variables with realizations in some set S.

On the basis of all players' actions a, the game is such that the realized allocation and monetary transfer is a realization[10] from the joint distribution of

$$(\widetilde{x}(a), \widetilde{t}(a)) = (\widetilde{x}_1(a), \widetilde{x}_2(a), \ldots, \widetilde{x}_N(a), \widetilde{t}_1(a), \widetilde{t}_2(a), \ldots, \widetilde{t}_N(a)) \in \Delta(\mathcal{O}),$$

where $\widetilde{x}_i(a)$ (resp., $\widetilde{t}_i(a)$) is a random variable that characterizes the distribution of allocations (resp., transfers) for player i given that the players take actions a.

These distributions characterizing the allocations and transfers are part of the specification of the game rules. The variable x_i (resp., t_i) is player i's realized allocation (resp., transfer) in its utility function. If $(\widetilde{x}_1(a), \widetilde{x}_2(a), \ldots, \widetilde{x}_N(a), \widetilde{t}_1(a), \widetilde{t}_2(a), \ldots, \widetilde{t}_N(a))$ is a degenerate random variable, then the allocation and transfer is deterministic when the players take actions a. As a function of all players' actions, the *expected* allocation to player i is $\overline{x}_i(a) = E(\widetilde{x}_i(a))$ and the *expected* transfer from player i is $\overline{t}_i(a) = E(\widetilde{t}_i(a))$.

Per the standard assumption from the economic theory literature that the game is common knowledge amongst the players, the players know the distributions of $(\widetilde{x}(\cdot), \widetilde{t}(\cdot))$. In other words, the players know the "rules" of the game.

The identification results can apply regardless of whether or not the econometrician knows the distributions of $(\widetilde{x}(\cdot), \widetilde{t}(\cdot))$, and/or the *expected* allocations and transfers $(\overline{x}(\cdot), \overline{t}(\cdot))$. In particular, any "randomness" that underlies non-deterministic allocations and transfers need not be explicitly modeled or known by the econometrician. Intuitively, if the econometrician does not know these objects, then it is possible to use the data to identify these objects. If the identification strategy depended on the specific game or the specific details of the game like the details of the allocation and transfer "rules," then different identification strategies would potentially be needed for each specification of how the allocations and transfers are determined. Of course, even if the econometrician does know the complete model of the game, the identification problem of recovering valuations from the data remains. Examples of an incomplete model of the game are discussed in Section 2.4. The fact that these features of the game can be identified can be useful by itself, depending on the application,

[10] By construction, these realizations are draws from the joint distribution and therefore by construction are independent from all other model quantities (e.g., the valuations of the players). This condition formalizes the notion that the allocation and transfer "don't depend on" anything except the actions of the players, and is (often implicitly) a standard condition in the related economic theory literature. Of course, the realized allocation and transfer will *indirectly* depend on the players' valuations, since the players' valuations determine the players' actions and the players' actions determine the realized allocation and transfer. For example, in the case of a tie for high bid in an auction, the auctioneer could flip a coin to determine who wins, but the outcome of the coin flip cannot somehow be "correlated" with the valuations of the players.

either as an object of interest by itself or as an "ingredient" in an analysis alongside the distribution of valuations.

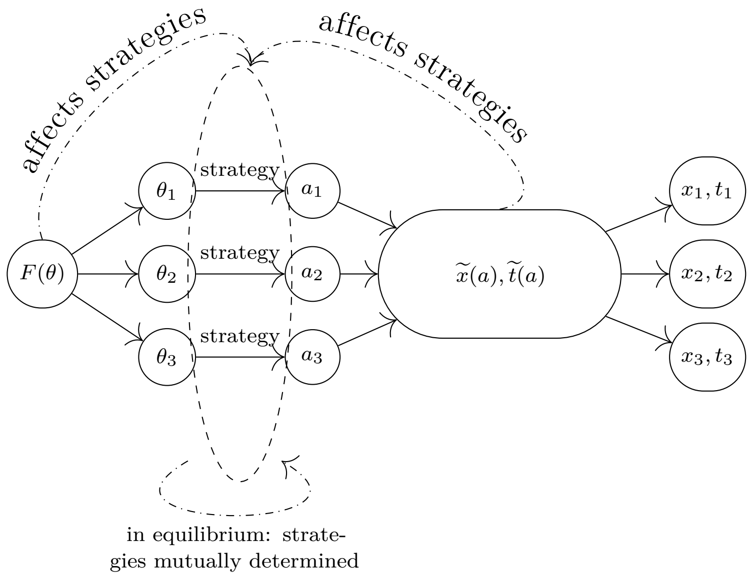

FIGURE 1. Graphical summary of game in the case of $N = 3$.

2.4. **Diagram of game framework.** Figure 1 provides a sketch of the basic idea of the incomplete information game framework studied in this paper. The game determines the allocations and monetary transfers (the x and t variables) on the basis of the actions of the players (the a variables). The strategy of player i determines the action a_i taken by player i as a function of the realized valuation θ_i of player i. The strategies depend implicitly on the rules of the game. In equilibrium, the strategies also depend on the strategies used by the other players, in the sense of mutual best responses. Obviously, as illustrated via specific examples in Appendix B, many economic environments can be modeled using this incomplete information game framework. This includes contests, auctions, procurement auctions and related models of oligopoly competition, strategic (non-"price taking") market behavior, partnership dissolution, and public good provision. The results apply to this class of incomplete information games, and therefore do not rely on specifics of particular examples. The range of examples in Appendix B shows the generality of this incomplete information game framework. The cited references in the examples include a range of results on equilibrium existence, as well as additional theoretical analysis of the models that can be used to motivate the assumptions used in the identification analysis.

2.5. **Data and identification problem.** The identification problem concerns recovering the distribution of valuations from observing many instances ("plays") of the game. For context, the related literature on identification in auctions has typically considered this identification problem in the case of auctions specifically. Variables relating to the actions, allocations, and transfers in *upper-case letters* represent quantities in the data, whereas quantities in *lower-case letters* represent variables in the underlying game. For example, A_i is the realized action in the data from player i, whereas a_i is the action variable in the underlying game from player i. Therefore, from each play of the game, the realized actions are $A = (A_1, A_2, \ldots, A_N)$, the realized allocations are $X = (X_1, X_2, \ldots, X_N)$, and the realized transfers are $T = (T_1, T_2, \ldots, T_N)$. Unless otherwise stated, the econometrician observes population data on the actions, allocations, and transfers. Hence, unless otherwise stated, the population data is $P(A, X, T)$. The realized allocations and realized transfers are linked to the realized actions through the game: in each instance of the game, by definition (X, T) is a draw from $(\widetilde{x}(A), \widetilde{t}(A)) = (\widetilde{x}_1(A), \widetilde{x}_2(A), \ldots, \widetilde{x}_N(A), \widetilde{t}_1(A), \widetilde{t}_2(A), \ldots, \widetilde{t}_N(A))$, the possibly non-deterministic allocation and transfer distributions given action profile A of the players. In the case of deterministic allocation and deterministic transfer, for a particular action profile A, then it can be understood that simply $X = \widetilde{x}(A) = (\widetilde{x}_1(A), \widetilde{x}_2(A), \ldots, \widetilde{x}_N(A))$ and $T = \widetilde{t}(A) = (\widetilde{t}_1(A), \widetilde{t}_2(A), \ldots, \widetilde{t}_N(A))$.

As detailed in the context of Lemma 1, the identification strategy can be based on less than full data on $P(A, X, T)$. If the econometrician specifies a complete model of the game, then the identification strategy can be based on only $P(A)$, essentially because in that case knowing A is enough to "know enough" about X and T. If the game involves a "two-part transfer," as in an auction with a participation cost, then the identification strategy can in certain cases be based on data from only one part of the transfer.

2.6. **Example.** This section provides an example of the incomplete information game framework studied in this paper. The example relates to models of strategic (non-"price taking") market behavior, specifically models based on multilateral double auctions. Further examples of this incomplete information game framework are provided in Appendix B. This example will illustrate, in particular, dependent valuations, monotone equilibrium, and partial identification.

These models involve N_s sellers (i.e., players that currently each own a unit of the object) and N_b buyers (i.e., players that potentially would each like to buy a unit of the object). The buyers and sellers interact in order to trade units of the object in exchange for monetary payments. The economic theory of such models has been developed in Chatterjee and Samuelson (1983), Myerson and Satterthwaite (1983), and Wilson (1985), in addition to a huge subsequent literature.[11] See Bolton and Dewatripont (2005, Chapter 7) for a textbook treatment.[12] The case of $N_s = 1 = N_b$ has seen particular attention, as models of bilateral trade.[13] The case of $N_s > 1$ and $N_b > 1$ has also seen particular attention, as "strategic" versions of supply and demand models, in which individual market participants do not act as competitive price takers. Although the theory literature has tended to treat these two cases separately, the identification strategy can accommodate both cases.

The valuation of player i for a unit of the object is the private information θ_i. The assumption that valuations are independent across players is implausible in many applications, both specifically in this example and in the incomplete information game framework studied in this paper more generally, for reasons that are broadly related to concerns about "correlated unobservables" in other structural models of market behavior in industrial organization, including the literature on identification of entry

[11] See Fudenberg, Mobius, and Szeidl (2007), Kadan (2007), or Araujo and De Castro (2009) for recent results.

[12] For monotonicity in the equilibrium strategies, see e.g., Chatterjee and Samuelson (1983, Theorem 1) and Satterthwaite and Williams (1989a, Definition of "regular" equilibrium) and Fudenberg, Mobius, and Szeidl (2007, Theorem 1).

[13] There are a variety of different "bilateral trade" or "bargaining" models, not all of which proceed in the same way. For example, Merlo and Tang (2012) study identification of a different bargaining model that evidently does not fit this incomplete information game framework.

models. Specifically, independent valuations would mean that a participant in a particular instance of a "market" would be unable to infer anything about the valuations of the other participants in that same instance based on its own valuation, so it is important that the identification strategy accommodates the possibility that the valuations are dependent/correlated across players.

The buyers announce "bid prices" and the sellers announce "ask prices" and trade proceeds. Suppose that $a_{(N_s)}$ is the N_s-th highest announcement and $a_{(N_s+1)}$ is the N_s+1-st highest announcement, both amongst the combined set of announcements (i.e., bids and asks) from buyers and sellers. Let $z(a) = ka_{(N_s)} + (1-k)a_{(N_s+1)}$ be the resulting transaction price, where $k \in [0,1]$ is a parameter of the model that might either be known or unknown by the econometrician (an example of a possibly incomplete specification of the model of the game). Then one possible allocation rule and transfer rule is

$$\overline{x}_i(a) = \begin{cases} 1 & \text{if } a_i > z(a) \\ p_i(a) & \text{if } a_i = z(a) \\ 0 & \text{if } a_i < z(a) \end{cases} \text{ and } \overline{t}_i(a) = \begin{cases} z(a) & \text{if i is a buyer and } a_i > z(a) \\ -z(a) & \text{if i is a seller and } a_i < z(a) \\ p_i(a)z(a) & \text{if i is a buyer and } a_i = z(a) \\ -(1-p_i(a))z(a) & \text{if i is a seller and } a_i = z(a) \\ 0 & \text{otherwise,} \end{cases}$$

where $p_i(a)$ reflects a tie-breaking rule with the condition that $\sum_{i=1}^{N} \overline{x}_i(a) = N_s$ for all a.[14] Therefore, ignoring ties by considering the generic situation that $a_{(N_s)} > a_{(N_s+1)}$, and because $a_{(N_s)} \geq z(a) \geq a_{(N_s+1)}$ with at least one inequality strict, the players with the N_s highest announcements, amongst both buyers and sellers, are allocated a unit of the object. The transaction price is $z(a)$, and buyers that are allocated a unit of the object pay $z(a)$ and sellers that are not allocated a unit of the object receive $z(a)$. See for example Fudenberg, Mobius, and Szeidl (2007) for more details. These allocation and transfer rules might be unknown by the econometrician, if the econometrician does not know k, in which case the identification strategy involves identifying the allocation and transfer rules directly from the data.

The identification strategy requires that the econometrician observe many instances of this process, as in observing many markets. Based on that data, the identification strategy recovers (bounds on) the joint distribution of valuations, which involves both the marginal distributions of the valuations of each participant and the dependence structure of the valuations across participants.

The main assumption of the identification strategy is that the players use monotone strategies. For buyers, this requires that buyers announce that they are willing to pay relatively more for a unit of the object when their valuation for a unit of the object is relatively higher. For sellers, this requires that sellers announce that they require a relatively higher payment for a unit of the object when their valuation for a unit of the object is relatively higher. As discussed further in Section 4, the assumption of a monotone strategy is particularly important in accommodating dependent valuations. Further, equilibrium strategies can be difficult to characterize (e.g., Leininger, Linhart, and Radner

[14] In particular, in the generic case of $a_{(N_s)} > a_{(N_s+1)}$, the tie-breaking rule is such that $p_i(a) = 1$ when $a_i = z(a)$ and $k = 1$ and $p_i(a) = 0$ when $a_i = z(a)$ and $k = 0$.

(1989) and Satterthwaite and Williams (1989a)), making it useful that assuming a property of the equilibrium is sufficient for the identification strategy, without needing to explicitly characterize the equilibrium solution. For example, in one particular case (with $k = 0$ and other assumptions), Satterthwaite and Williams (1989b) show that the equilibrium strategy for the buyers is the solution to a differential equation involving a combinatorial expression involving the unknown distribution of valuations.

These models also illustrate the importance of an emphasis on a partial identification strategy rather than a point identification strategy. Roughly, partial identification arises in games in which players use *weakly* but not *strictly* increasing strategies. Such "flat spots" in the strategy would arise for obvious reasons in games with discrete actions. For example, if there were an "entry cost" to participate in the market, and a corresponding discrete "participation" decision, then all buyers with a sufficiently low valuation would refuse to participate in the market and all sellers with a sufficiently high valuation would refuse to participate in the market. See also Example 2 in Appendix B. Such "flat spots" arise in the models in this section for more subtle reasons, even without any such discrete action. The use of such *weakly* but not *strictly* increasing strategies can be a feature of these models. Consider the special case of $N_s = 1 = N_b$. Chatterjee and Samuelson (1983, Example 2) show in a specific example that the strategy for the buyer or seller can involve a "flat spot" if the support of the distribution of valuations for the buyer is different from the support of the distribution of valuations for the seller.

Regardless of the reason for "flat spots" in the strategy, partial identification is the consequence because "flat spots" imply that a range of valuations lead to the use of the same action, so those valuations cannot be distinguished based on observed behavior. Therefore, insistence on a point identification strategy would rule out applications of the identification strategy to these models, and other models with other features that preclude point identification. Moreover, the issue of the possibility of "flat spots" in the strategies is exacerbated by the fact that the identification strategy accommodates dependent valuations. Specifically, the beliefs of players with different valuations are generically distinct even if they use the same action, so the identification strategy must account for the fact that players that use the same action do not necessarily have the same beliefs.

Based on the identification result for the distribution of valuations, the econometrician could then conduct any variety of subsequent steps of analysis. For example, in this particular example of a model of strategic market behavior, the econometrician could compute counterfactuals relating to manipulating the market structure (e.g, changing the number of buyers/sellers, imposing a tax, etc.).

2.7. **Definitions of stochastic ordering.** The identification strategy results in bounds on the multivariate distribution of valuations in terms of the usual multivariate stochastic order, which therefore concerns both the marginal distributions of each player's valuation and the dependence structure ("correlation") of the valuations.

Definition 1 (Upper set)**.** Let $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ and $y = (y_1, y_2, \ldots, y_d) \in \mathbb{R}^d$. A set $U \subseteq \mathbb{R}^d$ is an upper set if $x \in U$ and $y \geq x$ implies that $y \in U$. Per the standard, the condition $y \geq x$ is equivalent to $y_j \geq x_j$ for all $j = 1, 2, \ldots, d$.

Definition 2 (Usual multivariate stochastic order)**.** Let A and B be d-dimensional random vectors, with probability laws P_A and P_B. A is stochastically larger than B in the usual multivariate stochastic order if $P_A(U) \geq P_B(U)$ for all Borel measurable upper sets $U \subseteq \mathbb{R}^d$. And A is stochastically smaller than B in the usual multivariate stochastic order if B is stochastically larger than A in the usual multivariate stochastic order.

As formalized in Shaked and Shanthikumar (2007, Theorem 6.B.1), A is stochastically larger than B in the usual multivariate stochastic order exactly when there are $\hat{A}$ and $\hat{B}$ defined on the same probability space, such that $\hat{A}$ has the same distribution as A and $\hat{B}$ has the same distribution as B, and such that $\hat{A} \geq \hat{B}$ with probability 1. In the usual multivariate stochastic order, the partial identification result establishes that the random vector of valuations $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$ is stochastically larger than a certain random vector (i.e., "the distribution of θ is bounded below") and is stochastically smaller than another certain random vector (i.e., "the distribution of θ is bounded above"). The random vectors that are the upper and lower bounds for θ are themselves identified quantities, and have a constructive definition as a function of the observable data.

As discussed in Shaked and Shanthikumar (2007, Chapter 6), by the standard properties of the usual multivariate stochastic order, the partial identification result in terms of the usual multivariate stochastic order also implies partial identification of other quantities, including expectations of functions of the valuations and the multivariate cumulative distribution function of the valuations. In particular, the condition that the random vector A is stochastically larger than the random vector B in the usual multivariate stochastic order is equivalent to the condition that $E(\phi(A)) \geq E(\phi(B))$ for all weakly increasing functions ϕ for which the expectations exist.

In particular, because $\phi(X) = 1[X \leq t]$ is weakly decreasing in X, the condition that A with distribution function F_A is stochastically larger than B with distribution function F_B in the usual multivariate stochastic order implies that $F_A(t) \leq F_B(t)$ for all $t \in \mathbb{R}^d$.

As formalized in Definition 3, the condition that $F_A(t) \leq F_B(t)$ for all $t \in \mathbb{R}^d$ is known as the lower orthant order (e.g., Shaked and Shanthikumar (2007, Chapter 6.G.1)). The lower orthant order is a distinct sense of stochastic ordering. For random vectors, unlike for scalar random variables, the lower orthant ordering is implied by, but does not imply, the usual multivariate stochastic ordering. See Müller (2001) for more about the relationships between the senses of stochastic ordering when A and B are multivariate normal.

Definition 3 (Lower orthant stochastic order)**.** Let A and B be d-dimensional random vectors, with cumulative distribution functions F_A and F_B. A is stochastically larger than B in the lower orthant stochastic order if $F_A(t) \leq F_B(t)$ for all $t \in \mathbb{R}^d$. And A is stochastically smaller than B in the lower orthant stochastic order if B is stochastically larger than A in the lower orthant stochastic order.

Bounds on the distribution of valuations in the usual multivariate stochastic order also imply bounds on other quantities derived from the distribution of valuations, as discussed in Shaked and Shanthikumar (2007, Chapter 6). In their independent private values English auction setup, Haile and Tamer (2003) have shown how to use lower orthant bounds on the scalar distribution of valuations to bound the optimal reserve price in auctions.

2.8. **Baseline assumptions.** The following baseline assumptions are used. These assumptions are standard from the economic theory literature.

The players are assumed to be risk neutral, and therefore the *expected* allocations and transfers $\overline{x}_i(a)$ and $\overline{t}_i(a)$ determine *ex post* expected utility of player i as a function of its valuation and all players' actions:

$$\overline{U}_i(\theta_i, a) = \theta_i \overline{x}_i(a) - \overline{t}_i(a).$$

It holds that $\overline{U}_i(\theta_i, a)$ is the *ex post* expected utility because it depends on the actions of all players, which are not known *ex interim* by any individual player. In this paper, *ex post* expected utility refers to after the realization of the actions of all players in the game, which still can involve the expectation with respect to the non-degenerate randomness of the allocation rule and transfer rule.[15] *Ex interim* expected utility refers to before the realization of the actions of all players in the game, but after an individual player realizes its own valuation, which involves taking the expectation with respect to the player's beliefs about the other players' actions and the randomness of the allocation rule and transfer rule.

Because player i does not know the actions of the other players when it chooses its action, it must form beliefs about the actions of the other players. With dependent valuations, the beliefs held by player i about the actions of the other players depends on player i's realized valuation, so player i's beliefs are a distribution $\Pi_i(a_{-i}|\theta_i)$, defined over the actions of the other players, $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_N)$, that conditions on player i's realized valuation θ_i. In other words, with dependent valuations, players might be able to draw inferences about other players' valuations, and therefore other players' actions.

Independent valuations *Under Assumption 2 (Independent valuations), player i's beliefs are $\Pi_i(a_{-i})$, independent of player i's realized valuation. That is because with independent valuations, the realized valuation of player i does not revise the beliefs of player i about θ_{-i}, and therefore does not revise the beliefs of player i about a_{-i}.* ★

Therefore, *ex interim* expected utility of player i as a function of its valuation and its action is

$$V_i(\theta_i, a_i) = \theta_i E_{\Pi_i}(\overline{x}_i(a_i, a_{-i})|\theta_i) - E_{\Pi_i}(\overline{t}_i(a_i, a_{-i})|\theta_i).$$

With independent valuations, θ_i affects player i's *ex interim* expected utility only through the direct effect on the value of the object. With dependent valuations, θ_i also affects the expected allocation and expected transfer experienced by player i, even for a fixed action a_i, since player i's expected allocation and expected transfer depend on player i's beliefs about the other players' actions, and therefore on θ_i. This substantially complicates the identification problem under dependent valuations, compared to independent valuations.

Given this *ex interim* expected utility function, player i rationally takes an action that maximizes its *ex interim* expected utility given its realized valuation, so that its strategy $a_i(\theta_i)$ is supported on

[15]The utility that is actually realized (based on actually realized allocation and transfer) plays no role distinct from *ex post* expected utility.

the set of actions that maximizes *ex interim* expected utility:

$$a_i(\theta_i) \in \Delta(\arg\max_{a_i \in \mathcal{A}_i} V_i(\theta_i, a_i)). \tag{1}$$

Assumption 4 (Optimal strategy)**.** *For each $i \in \{1, 2, \ldots, N\}$, for each possible valuation θ_i, player i uses a strategy $a_i(\theta_i)$ when it has valuation θ_i, with $a_i(\theta_i) \in \Delta(\arg\max_{a_i \in \mathcal{A}_i} V_i(\theta_i, a_i))$, so each action taken according to the strategy $a_i(\theta_i)$ maximizes* ex interim *expected utility.*

In this assumption and other places, "possible valuation" means a valuation that is possible according to the (unknown) distribution of valuations. This assumption means that player i is rational, in the sense that it uses a strategy that maximizes its utility given its beliefs. Assumption 4 does not state that player i has correct beliefs. Instead, the subsequent Assumption 5 states that player i has correct beliefs. Also, Assumption 4 allows the use of a mixed strategy, but the identification strategy is based on the assumption of monotone equilibrium in monotone pure strategies, as formalized and discussed subsequently in Assumption 6. Breaking up the assumptions makes it easier to explain the identification strategy, by making it easier to refer to separate roles of the assumptions of using an optimal strategy, correct beliefs, and monotone equilibrium.

Let $P(A, X, T, \theta)$ be the "infeasible" data, regardless of whether those variables are observed by the econometrician. Then let $P(A_{-i}|\theta_i)$ be the realized distribution in the "infeasible" data over $A_{-i} = (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_N)$ conditional on the realized valuation θ_i of player i. Of course, θ_i is not observed by the econometrician, so the econometrician cannot condition on θ_i. In a Bayes Nash equilibrium, each player's beliefs are correct and correspond to the actual distribution of actions of the other players, in the sense that, for each player i, $\Pi_i(a_{-i} \in B|\theta_i) = P(A_{-i} \in B|\theta_i)$ for all Borel sets B. In other words, the beliefs of player i about a_{-i} when player i has valuation θ_i is equal to the actual realized distribution of A_{-i} when player i has valuation θ_i. This is the standard definition of correct beliefs with incomplete information.

Assumption 5 (Correct beliefs)**.** *For each $i \in \{1, 2, \ldots, N\}$, player i has correct beliefs, in the sense that, for each possible valuation θ_i, $\Pi_i(a_{-i} \in B|\theta_i) = P(A_{-i} \in B|\theta_i)$ for all Borel sets B.*

Independent valuations *Under Assumption 2 (Independent valuations), the assumption of correct beliefs is $\Pi_i(a_{-i} \in B) = P(A_{-i} \in B)$, since then beliefs do not depend on θ_i.* ★

As in other incomplete information setups, this assumption of correct beliefs implicitly supposes the realized distribution of actions (i.e., the data) comes from a single equilibrium corresponding to the players' beliefs. If multiple equilibria were played in the data, even with "correct beliefs" in each equilibrium, the realized distribution over actions in the data would be a mixture over the beliefs held by the player across equilibria, and thus the realized distribution over actions in the data would not equal players' beliefs. However, the econometrician need not have any *ex ante* knowledge of *which* equilibrium is selected in the case of multiple equilibria. If there is a unique equilibrium of the game, and indeed the economic theory literature has many results on equilibrium uniqueness, particularly but not only under the condition that the equilibrium is in monotone strategies as assumed in the identification strategy, then obviously the assumption that the data comes from a single equilibrium is automatically satisfied.

Under correct beliefs held by player i, $V_i(\theta_i, a_i) = \theta_i E_P(\overline{x}_i(a_i, A_{-i})|\theta_i) - E_P(\overline{t}_i(a_i, A_{-i})|\theta_i)$.

Appendix C shows that identification of some features of the distribution of valuations is robust to partial failures of these assumptions that all players use an optimal strategy with correct beliefs, basically as long as some players (but not necessarily all players) satisfy these assumptions.

3. Partial identification

The identification strategy is based around the utility maximization problem in Equation 1 facing each player as a function of its realized valuation. For each valuation θ_i of player i, let

$$\mathcal{A}_i^{s\theta}(\theta_i) = \{a_i^* \in \mathcal{A}_i : E_{\Pi_i}(\overline{x}_i(a_i, a_{-i})|\theta_i) \text{ and } E_{\Pi_i}(\overline{t}_i(a_i, a_{-i})|\theta_i) \text{ are differentiable functions of } a_i \text{ at } a_i = a_i^*\}.$$

Per Assumption 3 (Action space), $\mathcal{A}_i^{s\theta}(\theta_i) \subseteq \mathcal{A}_{i,\text{cont}}$, since a function cannot be differentiable at an isolated point of its domain. Derivatives with respect to a_i on the boundary of $\mathcal{A}_{i,\text{cont}}$ are one-sided derivatives.[16] The *ex interim* expected allocation $E_{\Pi_i}(\overline{x}_i(a_i, a_{-i})|\theta_i)$ and *ex interim* expected transfer $E_{\Pi_i}(\overline{t}_i(a_i, a_{-i})|\theta_i)$ tend to be differentiable functions of a_i, in part because the expectation with respect to player i's beliefs $\Pi_i(\cdot|\theta_i)$ is a smoothing operator.[17]

Differentiability is a standard but not universal condition in the economic theory literature on games with incomplete information. In such cases, $\mathcal{A}_i^{s\theta}(\theta_i) = \mathcal{A}_{i,\text{cont}}$ for all valuations θ_i. Differentiability is assumed for the point identification result in Section 4, but the partial identification result accommodates points of non-differentiability.

If player i with valuation θ_i takes an action $a_i \in \mathcal{A}_i^{s\theta}(\theta_i)$ according to the *ex interim* expected utility maximizing strategy from Assumption 4 (Optimal strategy), then the first order condition approach to an optimization problem implies a necessary condition.[18] Use the notation that $\text{int}(S)$ is the interior of some set S. Any *ex interim* expected utility maximizing action $\tilde{a}_i(\theta_i)$ such that $\tilde{a}_i(\theta_i) \in \text{int}(\mathcal{A}_i) \cap \mathcal{A}_i^{s\theta}(\theta_i)$ necessarily satisfies the condition that

$$\theta_i \left.\frac{\partial E_{\Pi_i}(\overline{x}_i(a_i, a_{-i})|\theta_i)}{\partial a_i}\right|_{a_i=\tilde{a}_i(\theta_i)} - \left.\frac{\partial E_{\Pi_i}(\overline{t}_i(a_i, a_{-i})|\theta_i)}{\partial a_i}\right|_{a_i=\tilde{a}_i(\theta_i)} = 0. \tag{2}$$

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[16] Specifically, the derivative on the lower bound of $\mathcal{A}_{i,\text{cont}}$ is the right derivative and the derivative on the upper bound of $\mathcal{A}_{i,\text{cont}}$ is the left derivative.

[17] For example, in auction models, if the auction format is such that the high bidder wins, and using Assumption 5 (Correct beliefs), then $E_{\Pi_i}(\overline{x}_i(a_i, a_{-i})|\theta_i) = P(\max_{j\neq i} A_j \leq a_i|\theta_i)$, which is generally a differentiable function of a_i across auction formats. This analysis uses that the event that two or more bidders tie for high bid has probability 0, as generally happens in auctions in equilibrium. The expected transfer is differentiable for similar reasons, for example in the case of a first price auction, $E_{\Pi_i}(\overline{t}_i(a_i, a_{-i})|\theta_i) = E_{\Pi_i}(a_i 1[\max_{j\neq i} A_j \leq a_i]|\theta_i) = a_i P(\max_{j\neq i} A_j \leq a_i|\theta_i)$.

[18] Because the use of one-sided derivatives on the boundary of the "continuous part" of the action space may be slightly unfamiliar, consider the first order conditions for the maximization problem $\max_{a_i\in\mathcal{A}_i} h(a_i)$. Suppose that $h(\cdot)$ has a right derivative at a_i^* and suppose that a_i^* is a local maximum. Since the right derivative exists at a_i^*, a_i^* cannot be the upper bound of $\mathcal{A}_{i,\text{cont}}$. Then, $\frac{h(t)-h(a_i^*)}{t-a_i^*} \leq 0$ for all $t > a_i^*$ in a neighborhood of a_i^*. Therefore, the right derivative at a_i^* must be non-positive. Suppose that $h(\cdot)$ has a left derivative at a_i^* and suppose that a_i^* is a local maximum. Since the left derivative exists at a_i^*, a_i^* cannot be the lower bound of $\mathcal{A}_{i,\text{cont}}$. Then, $\frac{h(t)-h(a_i^*)}{t-a_i^*} \geq 0$ for all $t < a_i^*$ in a neighborhood of a_i^*. Therefore, the left derivative at a_i^* must be non-negative. Therefore, if $h(\cdot)$ has a derivative at a_i^* and a_i^* is a local maximum, the right and left derivatives are the same, and equal to the usual derivative, which must therefore be zero.

If $\tilde{a}_i(\theta_i) = \alpha_i$ and $\tilde{a}_i(\theta_i) \in \mathcal{A}_i^{s\theta}(\theta_i)$, then $\tilde{a}_i(\theta_i)$ satisfies the condition[19] that

$$\theta_i \left. \frac{\partial E_{\Pi_i}(\overline{x}_i(a_i, a_{-i})|\theta_i)}{\partial a_i} \right|_{a_i = \tilde{a}_i(\theta_i)} - \left. \frac{\partial E_{\Pi_i}(\overline{t}_i(a_i, a_{-i})|\theta_i)}{\partial a_i} \right|_{a_i = \tilde{a}_i(\theta_i)} \leq 0. \tag{3}$$

And if $\tilde{a}_i(\theta_i) = \beta_i$ and $\tilde{a}_i(\theta_i) \in \mathcal{A}_i^{s\theta}(\theta_i)$, then $\tilde{a}_i(\theta_i)$ satisfies the condition that

$$\theta_i \left. \frac{\partial E_{\Pi_i}(\overline{x}_i(a_i, a_{-i})|\theta_i)}{\partial a_i} \right|_{a_i = \tilde{a}_i(\theta_i)} - \left. \frac{\partial E_{\Pi_i}(\overline{t}_i(a_i, a_{-i})|\theta_i)}{\partial a_i} \right|_{a_i = \tilde{a}_i(\theta_i)} \geq 0. \tag{4}$$

Equations 2-4 are useful for identification of the distribution of valuations, because these relationships connect the observed action taken by player i to the unobserved valuation of player i. However, these relationships depend on the unknown beliefs $\Pi_i(\cdot|\theta_i)$ of player i. Even under Assumption 5 (Correct beliefs), these beliefs are unknown because the valuation θ_i is unknown. Unknown beliefs can be dealt with by using the main assumption of the identification strategy, monotone equilibrium.

Assumption 6 (Weakly increasing strategy)**.** *For each $i \in \{1, 2, \ldots, N\}$, for each possible valuation θ_i, $a_i(\theta_i)$ is a pure strategy. And, for each $i \in \{1, 2, \ldots, N\}$, $a_i(\cdot)$ is a weakly increasing*[20] *function.*

The use of pure strategies implies that $a_i(\theta_i)$ is a particular action (i.e., a pure strategy) rather than a non-degenerate distribution (i.e., a mixed strategy). The economic theory literature has emphasized the importance of proving existence of equilibrium in monotone strategies.[21] General results establishing conditions for existence of pure strategy equilibria in monotone strategies include Maskin and Riley (2000), Athey (2001), McAdams (2003, 2006), and Reny (2011). Such results establish general conditions on the game that are sufficient for existence of monotone equilibrium. Moreover, again as cited elsewhere in this paper, particularly Appendix B, the economic theory literature has also established existence of pure strategy equilibria in monotone strategies in the context of specific games. Many economic theory papers establishing Assumption 6 assume affiliated valuations. Particularly in the context of affiliation in auctions, see Milgrom (2004, Section 5.4.1) for details. Further, many papers on identification in auctions assume affiliated valuations. The identification strategy in this paper does not require affiliation, as long as Assumption 6 is satisfied. Equilibria in monotone strategies can exist even without affiliated valuations, see for example Monteiro and Moreira (2006).

The assumption of monotone equilibrium is also intuitive. For example, in the context of the model of strategic market behavior discussed in Section 2.6, this assumption is that a buyer's bid is a weakly increasing function of that buyer's valuation and that a seller's ask is a weakly increasing function

[19]Recall derivatives on the boundary of $\mathcal{A}_{i,\text{cont}}$ are understood to be appropriate one-sided derivatives.

[20]It is straightforward to accommodate a weakly *decreasing* strategy, because a weakly *decreasing* strategy can be translated into a weakly *increasing* strategy by flipping the signs on the allocation rule and valuations, because if the strategy is weakly *decreasing* in the valuation θ_i, then the strategy is weakly *increasing* in the "negative valuation" $\hat{\theta}_i = -\theta_i$ with "negative allocation" $\hat{x}_i(a) = -\widetilde{x}_i(a)$. Note that $\hat{\theta}_i \hat{x}_i(a) = \theta_i \widetilde{x}_i(a)$ so utility is unaffected by flipping the signs in this way.

[21]Equilibrium existence in pure strategies is a general result for games with incomplete information. The economic theory (and existence) of such equilibria in pure strategies has been studied, for example, in Milgrom and Weber (1982, 1985), Dasgupta and Maskin (1986), Plum (1992), Reny (1999), Lizzeri and Persico (2000), Maskin and Riley (2003), and Jackson and Swinkels (2005) in addition to citations elsewhere in this paper, particularly Appendix B, amongst a huge literature.

of that seller's valuation. Appendix B provides a variety of other examples of games for which Assumption 6 is intuitive. For example, in applications to contests (Example 1), a monotone strategy simply requires the intuitive condition that players put forth effort as a weakly increasing function of their valuation for the object awarded by the contest. Or for another example, in applications to auctions (Example 2), a monotone strategy simply requires the intuitive condition that players make bids that are weakly increasing functions of their valuation for the object being auctioned.

Independent valuations *Under Assumption 2 (Independent valuations), it is possible to replace Assumption 6 with Assumption 7 stated directly on the game.*

Assumption 7 (Non-decreasing expected allocation rule)**.** *For each $i \in \{1, 2, \ldots, N\}$, $\overline{x}_i(a_i, a_{-i})$ is non-decreasing*[22] *in a_i for all a_{-i}.*

This is the standard[23] *condition that the expected allocation rule for player i is a non-decreasing function of the action of player i.* ★

Under Assumption 6, players use *weakly* increasing strategies, which accommodates the possibility that player i with valuation θ_i takes the action $a_i(\theta_i)$ and player i with valuation $\theta_i' \neq \theta_i$ also takes the same action $a_i(\theta_i') = a_i(\theta_i)$. Such "flat spots" in the strategy would necessarily arise for actions in $\mathcal{A}_{i,\text{disc}}$, the discrete part of the action space. An important example is a discrete "participation" decision relating to features like reserve prices or participation costs. Such "flat spots" in the strategy can also arise for more subtle reasons, even without discrete actions, as illustrated by the model of strategic market behavior in Section 2.6. Indeed, the economic theory literature on monotone equilibrium has tended to focus on *weakly* but not necessarily *strictly* increasing strategies. Therefore, it is important that the identification strategy assumes the use of *weakly* but not necessarily *strictly* increasing strategies.

These "flat spots" in the strategy result in partial identification, because "flat spots" imply that a range of valuations use the same action, so those valuations cannot be distinguished based on observed behavior. The main result in this paper is therefore a partial identification result. Insistence on a point identification strategy would rule out applications of the identification strategy to models with features like "flat spots" in the strategies that preclude point identification. Section 4 does provide stronger sufficient conditions for point identification. Moreover, the issue of the possibility of "flat spots" in the strategies is exacerbated by the fact that the identification strategy accommodates dependent valuations. Specifically, the beliefs of players with different valuations are generically distinct even if they use the same action, so the identification strategy must account for the fact that players that use the same action do not necessarily have the same beliefs.

[22]A non-increasing expected allocation rule can be translated into a non-decreasing expected allocation rule by "flipping" the sign of the action space, because if $\overline{x}_i(a_i, a_{-i})$ is non-increasing in a_i for all a_{-i}, then $\hat{x}_i(\hat{a}_i, a_{-i}) = \overline{x}_i(-\hat{a}_i, a_{-i})$ is non-decreasing in $\hat{a}_i$ for all a_{-i}. Assumption 7 implies $E_{\Pi_i}(\overline{x}_i(a_i, a_{-i}))$ is non-decreasing in a_i.

[23]For example, in contests from Example 1, $\overline{x}_i(a_i, a_{-i})$ is the "contest success function," and Assumption 7 states that the probability that player i wins the contest is a weakly increasing function of the effort of player i, holding fixed the effort of the other players. Standard contest success functions of the sort discussed in Example 1 have this property. Or, for example in auctions from Example 2, Assumption 7 states that the allocation to player i is a weakly increasing function of the bid of player i, holding fixed the bids of the other players. Standard auction formats in which the highest bid wins, resulting in functional forms for $\overline{x}_i(a_i, a_{-i})$ like in Example 2, have this property.

Under Assumption 6, the set $\{\theta_i : a_i(\theta_i) = a_i^*\}$ of valuations θ_i that use given action $a_i^* \in \mathcal{A}_i$ is necessarily an interval, which can be the empty set, a singleton set, or a non-degenerate interval (possibly infinite, and possibly including or not the endpoints).[24] Suppose that two distinct valuations θ_i and $\theta_i' \neq \theta_i$ both use the same action a_i^*. Then, all valuations between θ_i and θ_i' use that same action a_i^*. Under Assumption 1 (Dependent valuations), given that θ_i and θ_i' are in the support by construction, there is strictly positive mass of valuations between θ_i and θ_i'.[25] Therefore, a positive mass of valuations use the action a_i^*, and therefore there is a mass point in the observed distribution of actions at a_i^*. By the contrapositive, if there is not a mass point at some action a_i^*, then a_i^* is used by a unique valuation. This conclusion is not true without Assumption 6, since if $a_i(\cdot)$ were non-monotone, then the set $\{\theta_i : a_i(\theta_i) = a_i^*\}$ can be non-singleton, but not necessarily of positive probability under the distribution of θ_i. Therefore, if the strategy were non-monotone, then multiple valuations could use the same action a_i^* even though there is no mass point at a_i^*. Consequently, one role of Assumption 6 is to recover information about the beliefs of the players.

Let $\mathcal{A}_i^d$ be the support of A_i, the actions taken in the data by player i. And let

$$\tilde{\tilde{\mathcal{A}}}_i^d = \{a_i^* \in \mathcal{A}_i^d : \text{there is not a mass point at } a_i^* \text{ in the data}\} = \{a_i^* \in \mathcal{A}_i^d : P(A_i = a_i^*) = 0\}.$$

Obviously, the location of mass points is identified directly from the data. Under Assumption 6, for any $a_i^* \in \tilde{\tilde{\mathcal{A}}}_i^d$ there is a unique valuation θ_i^* that uses the action a_i^*, so conditioning on $A_i = a_i(\theta_i^*) = a_i^*$ is the same as conditioning on $\theta_i = \theta_i^*$. Therefore, $P(A_{-i} \in B|\theta_i = \theta_i^*) = P(A_{-i} \in B|A_i = a_i^*)$ for all Borel sets B. Consequently, under Assumption 5 (Correct beliefs), the beliefs of player i when it has valuation θ_i^* are equal to the distribution of $A_{-i}|(A_i = a_i(\theta_i^*) = a_i^*)$, the distribution of actions of the other players conditioning on player i taking its equilibrium action $a_i(\theta_i^*)$. In other words, the beliefs of a player observed to take an action $a_i^* \in \tilde{\tilde{\mathcal{A}}}_i^d$ are equal to the distribution in the data of the other players' actions conditional on player i taking action a_i^*. If multiple valuations used a_i^*, then the distribution in the data of the other players' actions conditional on player i taking action a_i^* would instead be a mixture over the beliefs held by player i with different valuations that use a_i^*.

For any realized action A_i, let

$$\mathcal{A}_i^{sA}(A_i) = \{a_i^* \in \mathcal{A}_i : E_P(\overline{x}_i(a_i, A_{-i})|A_i) \text{ and } E_P(\overline{t}_i(a_i, A_{-i})|A_i) \text{ are differentiable functions of } a_i \text{ at } a_i = a_i^*\}.$$

Per Assumption 3 (Action space), $\mathcal{A}_i^{sA}(A_i) \subseteq \mathcal{A}_{i,\text{cont}}$, since a function cannot be differentiable at an isolated point of its domain. Then, based on Equations 2-4, imposing Assumptions 5 (Correct beliefs) and 6 (Weakly increasing strategy), the optimality of an observed action $A_i = a_i(\theta_i)$ implies the following necessary conditions. If $A_i = a_i(\theta_i) \in \text{int}(\mathcal{A}_i) \cap \mathcal{A}_i^{sA}(A_i) \cap \tilde{\tilde{\mathcal{A}}}_i^d$, then $A_i = a_i(\theta_i)$ satisfies the condition that

$$\theta_i \left. \frac{\partial E_P(\overline{x}_i(a_i, A_{-i})|A_i)}{\partial a_i} \right|_{a_i = A_i} - \left. \frac{\partial E_P(\overline{t}_i(a_i, A_{-i})|A_i)}{\partial a_i} \right|_{a_i = A_i} = 0. \tag{5}$$

[24]Suppose $\{\theta_i : a_i(\theta_i) = a_i^*\}$ is not the empty set. And suppose that $a_i(\theta_i) = a_i^*$ and $a_i(\theta_i') = a_i^*$. Suppose without loss of generality that $\theta_i \leq \theta_i'$. Since $a_i(\cdot)$ is weakly increasing, any valuation between θ_i and θ_i' also uses action a_i^*.

[25]Because the support is convex by Assumption 1, there is some θ_i'' strictly between θ_i and θ_i' in the support of valuations. Any sufficiently small neighborhood of θ_i'' is also strictly between θ_i and θ_i', and by definition of support of a random variable, that neighborhood has positive mass under the distribution of valuations for player i.

If $A_i = a_i(\theta_i) = \alpha_i$ and $a_i(\theta_i) \in \mathcal{A}_i^{sA}(A_i) \cap \tilde{\tilde{\mathcal{A}}}_i^d$, then $A_i = a_i(\theta_i)$ satisfies the condition that

$$\theta_i \left. \frac{\partial E_P(\overline{x}_i(a_i, A_{-i})|A_i)}{\partial a_i} \right|_{a_i = A_i} - \left. \frac{\partial E_P(\overline{t}_i(a_i, A_{-i})|A_i)}{\partial a_i} \right|_{a_i = A_i} \leq 0. \tag{6}$$

And if $A_i = a_i(\theta_i) = \beta_i$ and $a_i(\theta_i) \in \mathcal{A}_i^{sA}(A_i) \cap \tilde{\tilde{\mathcal{A}}}_i^d$, then $A_i = a_i(\theta_i)$ satisfies the condition that

$$\theta_i \left. \frac{\partial E_P(\overline{x}_i(a_i, A_{-i})|A_i)}{\partial a_i} \right|_{a_i = A_i} - \left. \frac{\partial E_P(\overline{t}_i(a_i, A_{-i})|A_i)}{\partial a_i} \right|_{a_i = A_i} \geq 0. \tag{7}$$

Unlike Equations 2-4, Equations 5-7 do not involve the unknown beliefs of player i. Let

$$\Psi_i^x(z) \equiv \left. \frac{\partial E_P(\overline{x}_i(a_i, A_{-i})|A_i = z)}{\partial a_i} \right|_{a_i = z} \text{ and } \Psi_i^t(z) \equiv \left. \frac{\partial E_P(\overline{t}_i(a_i, A_{-i})|A_i = z)}{\partial a_i} \right|_{a_i = z} \tag{8}$$

and let

$$\Psi_i(z) \equiv \frac{\Psi_i^t(z)}{\Psi_i^x(z)}. \tag{9}$$

Then, rewriting Equations 5-7: if $A_i = a_i(\theta_i) \in \text{int}(\mathcal{A}_i) \cap \mathcal{A}_i^{sA}(A_i) \cap \tilde{\tilde{\mathcal{A}}}_i^d$, and $\Psi_i^x(A_i) \neq 0$, then

$$\theta_i = \Psi_i(A_i). \tag{10}$$

If $A_i = a_i(\theta_i) = \alpha_i$ and $A_i \in \mathcal{A}_i^{sA}(A_i) \cap \tilde{\tilde{\mathcal{A}}}_i^d$ and $\Psi_i^x(A_i) > 0$, so expected allocation is increasing[26] in action, then

$$\theta_i \leq \Psi_i(A_i). \tag{11}$$

If $A_i = a_i(\theta_i) = \beta_i$ and $A_i \in \mathcal{A}_i^{sA}(A_i) \cap \tilde{\tilde{\mathcal{A}}}_i^d$ and $\Psi_i^x(A_i) > 0$, then

$$\theta_i \geq \Psi_i(A_i). \tag{12}$$

Based on Equations 10-12, the partial identification result reflects the set of valuations that are compatible with the use of a given observed action. Therefore, structural identification of θ_i depends on identification of the functions $\Psi_i^x(\cdot)$ and $\Psi_i^t(\cdot)$.

Definition 4 (Action with game-structure identification)**.** An action $a_i \in \mathcal{A}_{i,\text{cont}}$ is an action with game-structure identification if $\Psi_i^x(a_i)$ and $\Psi_i^t(a_i)$ can be identified to exist, and $\Psi_i^x(a_i)$ and $\Psi_i^t(a_i)$ are point identified quantities. Per the convention, identification of derivatives on the boundary of $\mathcal{A}_{i,\text{cont}}$ is understood to concern identification of the corresponding *one-sided* derivative.

[26]For example, in contests it requires the intuitive condition that a player's probability of winning increases with the player's effort, and in auctions it requires the intuitive condition that a player's expected allocation increases with the player's bid. Since $\Psi_i^x(\cdot)$ is an identified function, the econometrician can check whether or not $\Psi_i^x(A_i) > 0$. If it happens that $\Psi_i^x(A_i) < 0$ instead, then similar bounds on θ_i obtain, flipping the direction of the inequalities. However, $\Psi_i^x(A_i) < 0$ is ruled out by Assumption 7 (Non-decreasing expected allocation rule). More generally, $\Psi_i^x(A_i) < 0$ seems to be at odds with the assumption of the use of a monotone increasing strategy (using "monotone comparative statics arguments") since the "cross-derivative" of *ex interim* expected utility with respect to (θ_i, a_i) would be negative evaluated in the case where $\Psi_i^x(A_i) < 0$. Therefore, to simplify the presentation, the results ignore the unlikely case that $\Psi_i^x(A_i) < 0$. Hence, the key restriction to apply these equations is that $\Psi_i^x(A_i) \neq 0$.

There are varied sources of game-structure identification. The term game-structure identification refers specifically to identification of the details of the structure of the game itself, specifically the functions $\Psi_i^x(\cdot)$ and $\Psi_i^t(\cdot)$, but not the distribution of valuations.

First, per Equation 8, the econometrician can have *ex ante* knowledge of $\overline{x}_i(\cdot)$ and $\overline{t}_i(\cdot)$ and take the expectation of those known functions with respect to the observed distribution of the actions in the data. Hence, game-structure identification obtains with a "standard" complete specification of the model, when the econometrician knows the allocation and transfer rules. Even in that case, part of the game-structure identification step is dealing with the beliefs of the players, using the data.

Second, because $\overline{x}_i(\cdot)$ is the expected allocation and $\overline{t}_i(\cdot)$ is the expected transfer,

(13)
$$\Psi_i^x(z) = \left.\frac{\partial E_P(E_P(X_i|A_i = a_i, A_{-i})|A_i = z)}{\partial a_i}\right|_{a_i=z} \text{ and } \Psi_i^t(z) = \left.\frac{\partial E_P(E_P(T_i|A_i = a_i, A_{-i})|A_i = z)}{\partial a_i}\right|_{a_i=z}$$

Per Equation 13, even without *ex ante* knowledge of the allocation and transfer rules, the econometrician can identify $\Psi_i^x(\cdot)$ and $\Psi_i^t(\cdot)$ by first identifying $\overline{x}_i(\cdot)$ and $\overline{t}_i(\cdot)$ based on the relationships $\overline{x}_i(a_i, a_{-i}) = E(\widetilde{x}_i(a_i, a_{-i})) = E(\widetilde{x}_i(a_i, a_{-i})|A_i = a_i, A_{-i} = a_{-i}) = E(X_i|A_i = a_i, A_{-i} = a_{-i})$ and $\overline{t}_i(a_i, a_{-i}) = E(\widetilde{t}_i(a_i, a_{-i})) = E(\widetilde{t}_i(a_i, a_{-i})|A_i = a_i, A_{-i} = a_{-i}) = E(T_i|A_i = a_i, A_{-i} = a_{-i})$. In each of these relationships, the first equality holds by definition of the expected allocation and transfer, the second equality holds since the randomness (if any) in the allocation and transfer is independent from the realized actions by construction, and the third equality holds by construction of the realized allocation and transfer. Hence, game-structure identification obtains even with an incomplete specification of the model, when the econometrician does not know the allocation and transfer rules.

Independent valuations *Under Assumption 2 (Independent valuations), the actions of different players are independent, so*

$$\Psi_i^x(z) = \Lambda_i^x(z) \equiv \left.\frac{\partial E_P(\overline{x}_i(a_i, A_{-i}))}{\partial a_i}\right|_{a_i=z} \text{ and } \Psi_i^t(z) = \Lambda_i^t(z) \equiv \left.\frac{\partial E_P(\overline{t}_i(a_i, A_{-i}))}{\partial a_i}\right|_{a_i=z}. \tag{14}$$

Also, let

$$\Lambda_i(z) \equiv \frac{\Lambda_i^t(z)}{\Lambda_i^x(z)}. \tag{15}$$

Then,[27]

$$\Lambda_i^x(z) = \left.\frac{\partial E_P(X_i|A_i = a_i)}{\partial a_i}\right|_{a_i=z} \text{ and } \Lambda_i^t(z) = \left.\frac{\partial E_P(T_i|A_i = a_i)}{\partial a_i}\right|_{a_i=z}. \tag{16}$$

Under Assumption 7 (Non-decreasing expected allocation rule), $\Lambda_i^x(z) \geq 0$ if $\Lambda_i^x(z)$ exists. Per Equations 14 and 16, under Assumption 2 (Independent valuations), the econometrician can identify $\Lambda_i^x(\cdot)$ and $\Lambda_i^t(\cdot)$ in "one-step," compared to the "two-steps" with dependent valuations. ★

[27]To establish the equalities in Equation 16, $E_P(X_i|A_i = a_i) = E_P(\widetilde{x}_i(A_i, A_{-i})|A_i = a_i) = E_P(\overline{x}_i(a_i, A_{-i})|A_i = a_i) = E_P(\overline{x}_i(a_i, A_{-i}))$, where the first equality holds by definition of the game (and resulting allocations), the second equality holds by standard properties of conditioning and the law of iterated expectations (with respect to any randomness in the allocation), and the third equality holds because the actions of different players are independent. It is similar for $E_P(T_i|A_i = a_i) = E_P(\overline{t}_i(a_i, A_{-i}))$.

The sufficient conditions for game-structure identification are formalized in Lemma 1 in Appendix A. The formalization is somewhat lengthy to state, but ultimately the conditions are weak. In short, Lemma 1 shows game-structure identification can be achieved either: (a) if data on allocations and transfers are observed, in which case relevant aspects of the allocation and transfer rules can be identified from the data even if those rules are not known *ex ante*, or (b) if the allocation and transfer rules are known *ex ante*, even if data on allocations and transfers are not observed. Depending on the application, either source of game-structure identification can be relevant.

Let

$$(17)\quad \tilde{\mathcal{A}}_i^d = \{a_i' \in \tilde{\tilde{\mathcal{A}}}_i^d \cap \mathcal{A}_{i,\text{cont}} : \Psi_i^x(a_i') \text{ exists and } \Psi_i^t(a_i') \text{ exists and } \Psi_i^x(a_i') > 0 \\ \text{and } a_i' \text{ is a point of game-structure identification per Definition 4}\},$$

By construction, $\tilde{\mathcal{A}}_i^d$ is an identified quantity. Moreover, it is possible to use economic theory to establish that certain actions a_i' satisfy the conditions of $\tilde{\mathcal{A}}_i^d$ under suitable regularity conditions on the game. Also let

$$(18)\qquad \rho_{Li}(a_i) = \{a_i' \in \tilde{\mathcal{A}}_i^d : \alpha_i < a_i' \leq a_i\} \text{ and } \rho_{Ui}(a_i) = \{a_i' \in \tilde{\mathcal{A}}_i^d : a_i \leq a_i' < \beta_i\}.$$

By construction, given any action $a_i' \in \rho_{Li}(a_i)$ that is not on the upper bound of $\mathcal{A}_{i,\text{cont}}$, the corresponding valuation compatible with using action a_i' is point identified by Equation 10. And if $a_i' \in \rho_{Li}(a_i)$ is on the upper bound of $\mathcal{A}_{i,\text{cont}}$, the corresponding valuation compatible with using action a_i' can be given a lower bound by Equation 12. Therefore, given any action $a_i' \in \rho_{Li}(a_i)$, the corresponding unobserved valuation compatible with using action a_i' is bounded below by $\Psi_i(a_i')$. Similarly, given any action $a_i' \in \rho_{Ui}(a_i)$, the corresponding unobserved valuation compatible with using action a_i' is bounded above by $\Psi_i(a_i')$.

Now, consider any $\tilde{\theta}_i < \Psi_i(a_i')$ with $a_i' \in \rho_{Li}(a_i)$. If θ_i' is any valuation consistent with using action a_i', then $\theta_i' \geq \Psi_i(a_i')$. Moreover, since $a_i' \in \mathcal{A}_i^d$ by construction, there is indeed some valuation θ_i' that uses action a_i'. By Assumption 6 (Weakly increasing strategy), the action used by valuation $\tilde{\theta}_i$ is weakly less than the action used by valuation $\theta_i' \geq \Psi_i(a_i') > \tilde{\theta}_i$, so the action used by valuation $\tilde{\theta}_i$ is weakly less than a_i'. Moreover, since $\tilde{\theta}_i \ngeq \Psi_i(a_i')$ by construction, valuation $\tilde{\theta}_i$ cannot use action a_i'. Consequently, player i with valuation $\tilde{\theta}_i$ must use an action strictly less than a_i'. By the contrapositive, any equilibrium action weakly greater than a_i' must correspond to a valuation weakly greater than $\Psi_i(a_i')$. Consequently, because $a_i' \leq a_i$, the valuation θ_i corresponding to the use of equilibrium action a_i must be weakly greater than $\Psi_i(a_i')$. Since the above holds for any $a_i' \in \rho_{Li}(a_i)$, the valuation θ_i corresponding to the use of equilibrium action a_i must be weakly greater than $\sup_{a_i' \in \rho_{Li}(a_i)} \Psi_i(a_i')$.

Consequently, $\sup_{a_i' \in \rho_{Li}(a_i)} \Psi_i(a_i')$ is a lower bound for the valuation corresponding to a_i. Similarly, $\inf_{a_i' \in \rho_{Ui}(a_i)} \Psi_i(a_i')$ is an upper bound for the valuation corresponding to a_i. It is possible to use economic theory to simplify these expressions under suitable regularity conditions on the game. In particular, under suitable regularity conditions on the game, $\Psi_i(\cdot)$ is a weakly increasing function on $\tilde{\mathcal{A}}_i^d$.[28] Under such conditions, if $a_{Li}^{**}(a_i) = \max(\rho_{Li}(a_i))$ exists, then $\sup_{a_i' \in \rho_{Li}(a_i)} \Psi_i(a_i') = \Psi_i(a_{Li}^{**}(a_i))$.

[28]Define $\mathcal{A}_i^\Psi = \{a_i : a_i \in \mathcal{A}_i^{sA}(a_i) \cap \tilde{\tilde{\mathcal{A}}}_i^d, \text{ and } \Psi_i^x(a_i) \neq 0\}$. Note that $\tilde{\mathcal{A}}_i^d \subseteq \mathcal{A}_i^\Psi$. Note that $\Psi_i(a_i)$ is a weakly increasing function of a_i for $a_i \in \text{int}(\mathcal{A}_i) \cap \mathcal{A}_i^\Psi$ by Equation 10, since the strategy is weakly increasing by Assumption 6 (Weakly increasing strategy). Note that $\Psi_i(a_i)$ is also defined for any $a_i \in \mathcal{A}_i^\Psi$, which can additionally potentially include the

And similarly, under such conditions, if $a_{Ui}^{**}(a_i) = \min(\rho_{Ui}(a_i))$ exists, then $\inf_{a_i' \in \rho_{Ui}(a_i)} \Psi_i(a_i') = \Psi_i(a_{Ui}^{**}(a_i))$.

Assumption 8 (Known bounds on valuations)**.** *For each $i \in \{1, 2, \ldots, N\}$, the valuation θ_i must be in the set $[\Theta_{Li}, \Theta_{Ui}]$.*

As often with partial identification results, the identified set can depend on *ex ante* known bounds on the partially identified quantity. By (heuristically) setting $\Theta_{Li} = -\infty$ and $\Theta_{Ui} = \infty$, it is possible to check the identification result without such known bounds. In many games, it might be reasonable to set $\Theta_{Li} = 0$, reflecting that the object has non-negative value to all players. Moreover, depending on the game, the partial identification result may depend on at most one of the lower or upper bound on the set of valuations. Assumption 8 is not the statement that the support of the valuations is $[\Theta_{Li}, \Theta_{Ui}]$, but rather is the statement that the support of the valuations is contained within $[\Theta_{Li}, \Theta_{Ui}]$. Hence, as also stated in Assumption 1 (Dependent valuations), the econometrician need not know the support of the valuations. Then, let

$$\kappa_{Li}(a_i) = \max\{\Theta_{Li}, \sup_{a_i' \in \rho_{Li}(a_i)} \Psi_i(a_i')\} \text{ and } \kappa_{Ui}(a_i) = \min\{\Theta_{Ui}, \inf_{a_i' \in \rho_{Ui}(a_i)} \Psi_i(a_i')\}. \tag{19}$$

Because the valuation must be between Θ_{Li} and Θ_{Ui}, it follows that the valuation corresponding to action a_i must be between $\kappa_{Li}(a_i)$ and $\kappa_{Ui}(a_i)$. Recall that $\rho_{Li}(a_i)$ and $\rho_{Ui}(a_i)$ are defined in Equation 18, and $\Psi_i(\cdot)$ is the identifiable function given in Equation 9 (see Lemma 1).

Independent valuations *Let*

$$\omega_{Li}(a_i) = \max\{\Theta_{Li}, \sup_{a_i' \in \rho_{Li}(a_i)} \Lambda_i(a_i')\} \text{ and } \omega_{Ui}(a_i) = \min\{\Theta_{Ui}, \inf_{a_i' \in \rho_{Ui}(a_i)} \Lambda_i(a_i')\} \tag{20}$$

Recall that $\rho_{Li}(a_i)$ and $\rho_{Ui}(a_i)$ are defined in Equation 18, and $\Lambda_i(\cdot)$ is the identifiable function given in Equation 15 (see Lemma 1). ★

Theorem 1. *Under Assumptions 1 (Dependent valuations), 3 (Action space), 4 (Optimal strategy), 5 (Correct beliefs), 6 (Weakly increasing strategy), and 8 (Known bounds on valuations) the distribution of valuations θ is partially identified, and the identification is constructive, because the distribution of θ is stochastically larger than the distribution of $(\kappa_{L1}(A_1), \kappa_{L2}(A_2), \ldots, \kappa_{LN}(A_N))$ and is stochastically smaller than the distribution of $(\kappa_{U1}(A_1), \kappa_{U2}(A_2), \ldots, \kappa_{UN}(A_N))$, in the sense of the usual multivariate stochastic order, where $(A_1, A_2, \ldots, A_N)$ is distributed according to the data $P(A, X, T)$ and $\kappa_{Li}(\cdot)$ and $\kappa_{Ui}(\cdot)$ are defined in Equation 19.*

Independent valuations *With independent valuations: replace Assumption 1 (Dependent valuations) with Assumption 2 (Independent valuations), replace Assumption 6 (Weakly increasing strategy)*

boundary of $\mathcal{A}_{i,\text{cont}}$. For example, consider the case $a_i = \alpha_i$ when indeed α_i is finite. If $\Psi_i(a_i)$ is a (right-)continuous function of a_i at $a_i = \alpha_i$ (and there is a corresponding interval $I_{\alpha,i}$ such that $I_{\alpha,i} \subseteq \mathcal{A}_i^{\Psi}$ with $\alpha_i \in I_{\alpha,i}$), then by standard arguments, $\Psi_i(a_i)$ is a weakly increasing function of a_i for $a_i \in (\text{int}(\mathcal{A}_i) \cap \mathcal{A}_i^{\Psi}) \cup \{\alpha_i\}$. Note that $\Psi_i(\alpha_i)$ exists because $\alpha_i \in \mathcal{A}_i^{\Psi}$ and $\Psi_i(\alpha_i) = \lim_{a_i' \to \alpha_i} \Psi_i(a_i')$ where the limit is taken along a sequence in $I_{\alpha,i}$. Let $a_i \in \text{int}(\mathcal{A}_i) \cap \mathcal{A}_i^{\Psi}$. Then, since $\Psi_i(\cdot)$ is weakly increasing on $\text{int}(\mathcal{A}_i) \cap \mathcal{A}_i^{\Psi}$, for all $\alpha_i < a_i' \leq a_i$, $\Psi_i(a_i') \leq \Psi_i(a_i)$, and therefore the limit $\Psi_i(\alpha_i) \leq \Psi_i(a_i)$. And by similar arguments and conditions, $\Psi_i(a_i) \leq \Psi_i(\beta_i)$ for $a_i \leq \beta_i$. Hence, under such conditions, $\Psi_i(a_i)$ is a weakly increasing function on $\mathcal{A}_i^{\Psi}$.

with Assumption 7 (Non-decreasing expected allocation rule), replace the definition of $\tilde{\mathcal{A}}_i^d$ *from Equation 17 with* $\tilde{\mathcal{A}}_i^d = \{a_i' \in \mathcal{A}_i^d \cap \mathcal{A}_{i,cont} : \Lambda_i^x(a_i')$ *exists and* $\Lambda_i^t(a_i')$ *exists and* $\Lambda_i^x(a_i') > 0$ *and* a_i' *is a point of game-structure identification per Definition 4*$\}$*, and replace the* κ *functions with the* ω *functions defined in Equation 20.* ★

Remark 1 (Properties of the identified set)**.** Theorem 1 is "partial-point" identification because some features of the distribution of valuations are point identified. Specifically, consider $\theta^* = (\theta_1^*, \theta_2^*, \ldots, \theta_N^*)$ from the distribution of valuations $F(\theta)$, with associated action profile $a(\theta^*) = (a_1(\theta_1^*), a_2(\theta_2^*), \ldots, a_N(\theta_N^*))$ such that, for all players i:

(1) It holds that $a_i(\theta_i^*) \in \text{int}(\mathcal{A}_{i,\text{cont}})$.
(2) And, there is not a point mass in the distribution of A_i at $a_i(\theta_i^*)$.
(3) And, $\Psi_i^x(a_i(\theta_i^*))$ and $\Psi_i^t(a_i(\theta_i^*))$ exist, and $\Psi_i^x(a_i(\theta_i^*)) > 0$.
(4) And, $a_i(\theta_i^*)$ is an action with game-structure identification per Definition 4.

Under these conditions, $a_i(\theta_i^*) \in \rho_{Li}(a_i(\theta_i^*)) \cap \rho_{Ui}(a_i(\theta_i^*))$. Therefore, it must be that $\kappa_{Li}(a_i(\theta_i^*)) \geq \Psi_i(a_i(\theta_i^*))$ and $\kappa_{Ui}(a_i(\theta_i^*)) \leq \Psi_i(a_i(\theta_i^*))$, and therefore per the identification result, it must be that any valuation θ_i consistent with the use of action $a_i(\theta_i^*)$ is between $\Psi_i(a_i(\theta_i^*))$ and $\Psi_i(a_i(\theta_i^*))$, and therefore must equal $\Psi_i(a_i(\theta_i^*))$, and hence there is point identification of the valuation corresponding to the use of an action satisfying these conditions. These are essentially the assumptions used in the point identification result in Theorem 2.

There is informative partial identification as long as $\tilde{\mathcal{A}}_i^d \cap \text{int}(\mathcal{A}_{i,\text{cont}}) \neq \emptyset$. Informative partial identification means that the bounds depend non-trivially on the data, i.e., are informative even without *ex ante* bounds on valuations, in the heuristic limit of $\Theta_{Li} = -\infty$ and $\Theta_{Ui} = \infty$.[29] If so, then $\rho_{Li}(a_i) \cup \rho_{Ui}(a_i) \neq \emptyset$ for all actions a_i.[30] And if so, then at least one of "$\sup_{a_i' \in \rho_{Li}(a_i)}$" and/or "$\inf_{a_i' \in \rho_{Ui}(a_i)}$" appearing in Equation 19 is taken over a non-empty set. And if so, Theorem 1 results in informative partial identification. By inspecting the definition, $\tilde{\mathcal{A}}_i^d \cap \text{int}(\mathcal{A}_{i,\text{cont}}) = \emptyset$ if:

(1) The action space for player i is entirely discrete (i.e., $\mathcal{A}_{i,\text{cont}} = \emptyset$).
(2) Or, there are only mass points in the distribution of A_i (i.e., $\tilde{\tilde{\mathcal{A}}}_i^d = \emptyset$).
(3) Or, there are no actions a_i' for which $\Psi_i^x(a_i')$ and $\Psi_i^t(a_i')$ exist and $\Psi_i^x(a_i') > 0$.
(4) Or, there are no actions for player i with game-structure identification per Definition 4.

Appendix D develops partial identification results under an additional assumption that can apply even in those cases. Conversely, $\tilde{\mathcal{A}}_i^d \cap \text{int}(\mathcal{A}_{i,\text{cont}}) \neq \emptyset$ if there is an action a_i^* satisfying:

(1) $a_i^* \in \text{int}(\mathcal{A}_{i,\text{cont}})$.
(2) And, there is not a mass point at a_i^* in the distribution of A_i (i.e., $a_i^* \in \tilde{\tilde{\mathcal{A}}}_i^d$).
(3) And, $\Psi_i^x(a_i^*)$ and $\Psi_i^t(a_i^*)$ exist and $\Psi_i^x(a_i^*) > 0$.
(4) And, a_i^* is an action with game-structure identification per Definition 4.

[29]In extreme cases like $\Theta_{Li} = \Theta_{Ui}$, the data is irrelevant and identification comes entirely from Assumption 8.

[30]It holds that $\rho_{Li}(a_i) \cup \rho_{Ui}(a_i) = \{a_i' \in \tilde{\mathcal{A}}_i^d : \alpha_i < a_i' < \beta_i\}$ when $a_i \in \text{int}(\mathcal{A}_{i,\text{cont}})$ and $\rho_{Li}(a_i) \cup \rho_{Ui}(a_i) = \{a_i' \in \tilde{\mathcal{A}}_i^d : \alpha_i \leq a_i' < \beta_i\}$ when $a_i \leq \alpha_i$ and $\rho_{Li}(a_i) \cup \rho_{Ui}(a_i) = \{a_i' \in \tilde{\mathcal{A}}_i^d : \alpha_i < a_i' \leq \beta_i\}$ when $a_i \geq \beta_i$. If $\tilde{\mathcal{A}}_i^d \neq \emptyset$, then in particular $\mathcal{A}_{i,\text{cont}} \neq \emptyset$, and per Assumption 3, it must therefore be that $\alpha_i < \beta_i$ since $\alpha_i = \beta_i$ is not allowed. Therefore, any $a_i \in \tilde{\mathcal{A}}_i^d \cap \text{int}(\mathcal{A}_{i,\text{cont}})$ is in $\rho_{Li}(a_i) \cup \rho_{Ui}(a_i)$.

At least in general, excepting especially "non-smooth" games that either induce "pooling" at finitely many actions, or have everywhere non-differentiable *ex interim* expected allocation and/or *ex interim* expected transfer, it is reasonable to expect that such an action exists as long as the action space is not entirely discrete. Of course, that is an empirical question for any given application.

Finally, there is partial identification (not point identification) of the valuation corresponding to using an action on the boundary of $\mathcal{A}_{i,\text{cont}}$, or using an action in $\mathcal{A}_{i,\text{disc}}$. For example, if action a_i is on the lower bound of $\mathcal{A}_{i,\text{cont}}$ or if action a_i is in $\mathcal{A}_{i,\text{disc}}^{\text{low}}$, then $\rho_{Li}(a_i) = \emptyset$, so the lower bound on the valuation associated with taking such an action is Θ_{Li} from Assumption 8 (Known bounds on valuations). But the upper bound on the valuation associated with taking such an action concerns the infimum of the set of possible valuations corresponding to using an action in $\rho_{Ui}(a_i)$.

The identified bounds for the valuation corresponding to the use of action $a_i \in \mathcal{A}_{i,\text{disc}}^{\text{low}}$ are the same as the identified bounds for the valuation corresponding to the use of action $a_i' \in \mathcal{A}_{i,\text{disc}}^{\text{low}}$. Similarly, the identified bounds for the valuation corresponding to the use of action $a_i \in \mathcal{A}_{i,\text{disc}}^{\text{high}}$ are the same as the identified bounds for the valuation corresponding to the use of action $a_i' \in \mathcal{A}_{i,\text{disc}}^{\text{high}}$. In many games, $|\mathcal{A}_{i,\text{disc}}^{\text{low}}| \leq 1$ and $|\mathcal{A}_{i,\text{disc}}^{\text{high}}| \leq 1$, in which case this observation about the identification result is irrelevant. However, in other games, $|\mathcal{A}_{i,\text{disc}}^{\text{low}}| > 1$ and/or $|\mathcal{A}_{i,\text{disc}}^{\text{high}}| > 1$. In particular, some games may have entirely discrete action spaces, so that $|\mathcal{A}_{i,\text{disc}}^{\text{low}} \cup \mathcal{A}_{i,\text{disc}}^{\text{high}}|$ is the total number of actions. Appendix D develops an extension of the identification strategy that is useful in games with many discrete actions, or entirely discrete action spaces. Based on that identification strategy, the identified bounds for the valuations can differ for different actions in $\mathcal{A}_{i,\text{disc}}^{\text{low}}$ and for different actions in $\mathcal{A}_{i,\text{disc}}^{\text{high}}$. Essentially, that extension is a "discrete analogue" of the identification strategy in this section. The extension is complicated by the need to recover information about the beliefs of the players, when *each* action is used by a range of valuations, so the extension involves an extra step of "bounding" the beliefs of the players.

4. Point identification

As noted after Theorem 1, the partial identification result establishes point identification of features of the distribution of valuations satisfying certain conditions. If these conditions hold in general, as follows, then the entire distribution of valuations is point identified.

Assumption 9 (No use of discrete actions and no point masses in distribution of actions)**.** *For each* $i \in \{1, 2, \ldots, N\}$*,* $\mathcal{A}_i^d \subseteq \mathcal{A}_{i,cont}$ *and* $\tilde{\tilde{\mathcal{A}}}_i^d = \mathcal{A}_i^d$*.*

According to Theorem 1, any action in $\mathcal{A}_{i,\text{disc}}$ corresponds to partial identification of the corresponding valuation. Hence, Assumption 9 disallows the use of actions in $\mathcal{A}_{i,\text{disc}}$. Of course, that holds by construction if indeed the action space is entirely continuous, so that $\mathcal{A}_{i,\text{disc}} = \emptyset$. Further, if there are relatively fewer point masses in the distribution of the actions in the data, then the identified set for the distribution of valuations in Theorem 1 becomes tighter, all else equal, because the $\rho_{Li}(a_i)$ and $\rho_{Ui}(a_i)$ sets become larger. Hence, Assumption 9 (No use of discrete actions and no point masses in distribution of actions) is used in the point identification result. Obviously, this assumption can be checked in applications. Under Assumptions 1 (Dependent valuations) and 6 (Weakly increasing strategy), and the analysis of Section 3, the lack of point masses is equivalent to the condition that the strategy is *strictly* increasing.[31]

Assumption 10 (Differentiable *ex interim* expected allocation and transfer)**.** *For each $i \in \{1, 2, \dots, N\}$, there is a set $\mathcal{E}_{i,d}$ with $P(A_i \in \mathcal{E}_{i,d}) = 0$ such that the game and player i's beliefs are such that, for each possible valuation θ_i, the expected allocation $E_{\Pi_i}(\overline{x}_i(a_i, a_{-i})|\theta_i)$ and the expected transfer $E_{\Pi_i}(\overline{t}_i(a_i, a_{-i})|\theta_i)$ are differentiable functions of a_i, evaluated at any $a_i^* \in$ support$(a_i(\theta_i)) \cap \mathcal{E}_{i,d}^C$.*

Assumption 10 requires that *ex interim* expected allocation and *ex interim* expected transfer given valuation θ_i are differentiable on the support of the strategy $a_i(\theta_i)$. Of course, under Assumption 6 (Weakly increasing strategy), $a_i(\theta_i)$ is a degenerate random variable (i.e., a pure strategy), in which case $a_i^* = a_i(\theta_i)$. However, under Assumption 4 (Optimal strategy) alone, mixed strategies are allowed. As discussed above, breaking up the assumptions in this way makes it easier to refer to the separate roles of the assumptions. In many games, the *ex interim* expected allocation and *ex interim* expected transfer are differentiable on the entire action space. If *ex interim* expected allocation and *ex interim* expected transfer have relatively more points of differentiability, then the identified set for the distribution of valuations in Theorem 1 becomes tighter, all else equal, because the $\rho_{Li}(a_i)$ and $\rho_{Ui}(a_i)$ sets become larger. Hence, Assumption 10 (Differentiable *ex interim* expected allocation and transfer) is used in the point identification result. Assumption 10 allows a probability zero exceptional set of actions at which differentiability fails. Of course, the notation S^C for some set S is the complement of the set S.

Under Assumptions 1 (Dependent valuations), 5 (Correct beliefs), 6 (Weakly increasing strategy), and 9 (No use of discrete actions and no point masses in distribution of actions), Assumption 10 can be checked in applications as follows. Let θ_i be some possible valuation, and let $z_i = a_i(\theta_i)$ be the corresponding action from Assumption 6. By construction, per Assumption 6, $\{z_i\} = \text{support}(a_i(\theta_i))$. By the analysis of Section 3, since there is no point mass at z_i per Assumption 9, θ_i must be the unique valuation to use action z_i. Therefore, by Assumptions 5 and 6, and the analysis of Section 3, $\Psi_i^x(z_i) = \left.\frac{\partial E_{\Pi_i}(\overline{x}_i(a_i,a_{-i})|\theta_i)}{\partial a_i}\right|_{a_i=z_i}$ and $\Psi_i^t(z_i) = \left.\frac{\partial E_{\Pi_i}(\overline{t}_i(a_i,a_{-i})|\theta_i)}{\partial a_i}\right|_{a_i=z_i}$. Therefore, the differentiability condition in Assumption 10 is equivalent to existence of $\Psi_i^x(z_i)$ and $\Psi_i^t(z_i)$ for probability 1 of actions z_i used in the data. The identification problem of determining existence of $\Psi_i^x(z_i)$ and $\Psi_i^t(z_i)$ was discussed in the context of game-structure identification in Section 3.

Assumption 11 (Game-structure identification)**.** *For each $i \in \{1, 2, \dots, N\}$, there is a set $\mathcal{E}_{i,r}$ with $P(A_i \in \mathcal{E}_{i,r}) = 0$ such that if $a_i \in \mathcal{A}_i^d \cap \mathcal{E}_{i,r}^C$ is such that $\Psi_i^x(a_i)$ and $\Psi_i^t(a_i)$ exist then a_i is an action with game-structure identification per Definition 4.*

If there are relatively more actions with game-structure identification, then the identified set for the distribution of valuations in Theorem 1 becomes tighter, all else equal, because the $\rho_{Li}(a_i)$ and $\rho_{Ui}(a_i)$ sets become larger. Hence, Assumption 11 (Game-structure identification) is used in the point identification result. Assumption 11 requires game-structure identification for all actions used

[31]If two valuations use the same action, then there is a point mass at that action. So, if there are no point masses, then no two valuations use the same action, so the strategy must indeed be *strictly* increasing. Conversely, obviously if the strategy is *strictly* increasing, then there are no point masses in the distribution of actions by Assumption 1.

in the data except for the probability zero exceptional set $\mathcal{E}_{i,r}$. This accommodates the possibility that game-structure identification may fail on a set of probability zero, as discussed after Lemma 1. Assumptions 1, 5, 6, 9, and 10, and the analysis of Section 3, imply that $\Psi_i^x(a_i)$ and $\Psi_i^t(a_i)$ actually do exist for $a_i \in \mathcal{A}_i^d \cap \mathcal{E}_{i,d}^C$.

Assumption 12 (Non-zero marginal expected allocation)**.** *For each $i \in \{1, 2, \ldots, N\}$, there is a set $\mathcal{E}_{i,m}$ with $P(A_i \in \mathcal{E}_{i,m}) = 0$ such that $\Psi_i^x(a_i) \neq 0$ for $a_i \in \mathcal{A}_i^d \cap \mathcal{E}_{i,m}^C$.*

Independent valuations *Under Assumption 2 (Independent valuations), by Darboux's theorem, if $\Psi_i^x(z) = \Lambda_i^x(z) = \frac{\partial E_P(\overline{x}_i(a_i, A_{-i}))}{\partial a_i}\Big|_{a_i=z} \neq 0$ on an interval subset of $\mathcal{A}_i^d$, then $\Lambda_i^x(\cdot)$ is of constant sign on that interval, and hence $E_P(\overline{x}_i(a_i, A_{-i}))$ is a strictly monotone function of a_i on that interval, a sort of strengthened version of Assumption 7 (Non-decreasing expected allocation rule).*

If there are relatively more actions a_i' such that $\Psi_i^x(a_i') \neq 0$, then the identified set for the distribution of valuations in Theorem 1 becomes tighter, all else equal, because the $\rho_{Li}(a_i)$ and $\rho_{Ui}(a_i)$ sets become larger. Hence, Assumptions 12 (Non-zero marginal expected allocation) is used in the point identification result. Assumption 12 can be checked in applications since $\Psi_i^x(\cdot)$ is an identified function, per Assumption 11. Assumption 12 allows a probability zero exceptional set $\mathcal{E}_{i,m}$.

As a technical note, point identification is achieved in the identification strategy by applying Equation 2 to a particular action in order to recover the corresponding valuation, except for actions in the probability zero exceptional set of actions $\mathcal{E} = \prod_i \mathcal{E}_i$ with $\mathcal{E}_i = (\text{int}(\mathcal{A}_{i,\text{cont}}))^C \cup \mathcal{E}_{i,d} \cup \mathcal{E}_{i,r} \cup \mathcal{E}_{i,m}$. In other words, considering the "unobserved" joint distribution of θ and A, $P(\theta, A)$, it is possible to write that $P(\theta \in B) = P(\theta \in B, A \in \mathcal{E}^C) + P(\theta \in B, A \in \mathcal{E}) = P(\theta \in B, A \in \mathcal{E}^C) = P(\theta \in B | A \in \mathcal{E}^C)$ for any Borel set B, and hence it is enough to restrict the identification problem to recovering the distribution of θ from actions in the probability 1 subset of actions $\mathcal{E}^C$.

Theorem 2. *Under Assumptions 1 (Dependent valuations), 3 (Action space), 4 (Optimal strategy), 5 (Correct beliefs), 6 (Weakly increasing strategy), 9 (No use of discrete actions and no point masses in distribution of actions), 10 (Differentiable* ex interim *expected allocation and transfer), 11 (Game-structure identification), and 12 (Non-zero marginal expected allocation), the distribution of valuations θ is point identified, and the identification is constructive, because the distribution of θ equals the distribution of $(\Psi_1(A_1), \Psi_2(A_2), \ldots, \Psi_N(A_N))$, where $(A_1, A_2, \ldots, A_N)$ is distributed according to the data $P(A, X, T)$ and $\Psi_i(\cdot)$ is the identifiable function given in Equation 9 (see Lemma 1).*

Independent valuations *With independent valuations: replace Assumption 1 (Dependent valuations) with Assumption 2 (Independent valuations), drop Assumption 6 (Weakly increasing strategy) and replace the Ψ functions with the Λ functions defined in Equation 15.* ★

As a point identification result, Theorem 2 drops Assumption 8 (Known bounds on valuations), compared to Theorem 1. In short, Theorem 2 shows sufficient conditions under which it is possible to point identify the distribution of valuations. The partial identification result can deliver point identification of features of the distribution of valuations satisfying essentially the assumptions discussed in this section. Consequently, if a large probability mass of actions satisfies these assumptions, then the partial identification result is "close" to point identification, in the sense that the large probability mass of actions that satisfies these assumptions would result in point identification of the corresponding valuations.

5. Conclusions

This paper develops identification results for the distribution of valuations in a class of incomplete information games that determine an allocation of units of a valuable object and arrangement of monetary transfers on the basis of the actions taken by the players. The identification results are constructive and are based on the assumption of monotone equilibrium, which is particularly important in the fact that the identification strategy accommodates dependent valuations. The identification results can flexibly deliver either point identification or partial identification. The fact that the identification result is primarily a partial identification result is important, because insistence on a point identification strategy would preclude many applications of the identification strategy to games which partially identify the distribution of valuations.

The appendices include further discussion and some extensions of the identification results. Appendix A provides sufficient conditions for game-structure identification. Appendix B provides further discussion of specific examples of the incomplete information game framework studied in this paper. Appendix C shows that identification of some features of the distribution of valuations are robust to partial failures of the equilibrium assumption. Appendix D shows that the identification strategy can be extended under an additional assumption, which is especially useful to handle situations involving an entirely discrete action space. This extension involves “bounding” the beliefs of the players.

Appendix A. Sufficient conditions for game-structure identification

Lemma 1 (Sufficient conditions for game-structure identification)**.** *Suppose that Assumptions 1 (Dependent valuations) and 3 (Action space) are satisfied. Let an action $a_i \in \mathcal{A}_{i,cont}$ be given, and suppose that one of the following conditions is true.*

(1) [Two-step game-structure identification] *It holds that $a_i \in \mathcal{A}_i^d$, and there is a set $\mathcal{S}$ containing a_i such that $\mathcal{A}_i^d \cap \mathcal{A}_{i,cont} \cap \mathcal{S}$ is a non-degenerate interval and such that the econometrician can point identify the conditional expectations $E_P(X_i|A_i = a_i', A_{-i} = a_{-i})$ and $E_P(T_i|A_i = a_i', A_{-i} = a_{-i})$ for all $a_i' \in \mathcal{A}_i^d \cap \mathcal{A}_{i,cont} \cap \mathcal{S}$ and $a_{-i} \in \tilde{\mathcal{A}}_{-i}^d(a_i')$, where $\tilde{\mathcal{A}}_{-i}^d(a_i')$ has probability 1 according to the distribution $A_{-i}|(A_i = a_i)$. The distribution $A_{-i}|(A_i = a_i)$ is point identified. If $a_i \in int(\mathcal{A}_{i,cont})$, then $a_i \in int(\mathcal{A}_i^d \cap \mathcal{A}_{i,cont} \cap \mathcal{S})$. The data is $P(A, X, T)$.*

(2) [Two-step game-structure identification II] *Assumption 6 (Weakly increasing strategy) is satisfied. It holds that $a_i \in (int(\mathcal{A}_{i,cont}) \cap bd(\mathcal{A}_i^d \cap \mathcal{A}_{i,cont}))^C \cap (\mathcal{A}_i^d \cap \mathcal{A}_{i,cont})$. Also, it holds that $a_i \in \tilde{\tilde{\mathcal{A}}}_i^d$ and there is an interval $\mathcal{I}$ containing a_i such that $\tilde{\tilde{\mathcal{A}}}_i^d \cap \mathcal{A}_{i,cont} \cap \mathcal{I}$ is a non-degenerate interval. There is a neighborhood $\mathcal{N}$ of a_i such that $\overline{x}_i(a_i', a_{-i})$ and $\overline{t}_i(a_i', a_{-i})$ are continuous*[32] *functions, at all $a_i' \in \mathcal{A}_{i,cont} \cap \mathcal{N}$ and $a_{-i} \in \tilde{\mathcal{A}}_{-i}^d(a_i')$, where $\tilde{\mathcal{A}}_{-i}^d(a_i')$ has probability 1 according to the distribution $A_{-i}|(A_i = a_i)$. The conditional distribution $\theta_{-i}|\theta_i$ is continuous in θ_i at θ_i^*, where θ_i^* is the unique valuation to use $a_i \in \tilde{\tilde{\mathcal{A}}}_i^d$, in the sense that if $\theta_i' \to \theta_i^*$ then the*

[32] Of course, continuity is defined with respect to the domain $\prod_i \mathcal{A}_i$.

conditional density $f(\theta_{-i}|\theta_i = \theta_i')$ *converges to the conditional density* $f(\theta_{-i}|\theta_i = \theta_i^*)$ *for all* θ_{-i}.[33] *The data is* $P(A, X, T)$.

(3) [One-step game-structure identification] *Assumption 2 (Independent valuations) is satisfied. It holds that* $a_i \in \mathcal{A}_i^d$ *is such that there is a set* $\mathcal{S}$ *containing* a_i *such that* $\mathcal{A}_i^d \cap \mathcal{A}_{i,cont} \cap \mathcal{S}$ *is a non-degenerate interval, such that the econometrician can point identify* $E_P(X_i|A_i = a_i')$ *and* $E_P(T_i|A_i = a_i')$ *for all* $a_i' \in \mathcal{A}_i^d \cap \mathcal{A}_{i,cont} \cap \mathcal{S}$. *If* $a_i \in int(\mathcal{A}_{i,cont})$, *then* $a_i \in int(\mathcal{A}_i^d \cap \mathcal{A}_{i,cont} \cap \mathcal{S})$. *The data is* $P(A, X, T)$.

(4) [*Ex ante* known allocation and transfer rules] *It holds that* $a_i \in \mathcal{A}_i^d$, *and there is a set* $\mathcal{S}$ *containing* a_i *such that* $\mathcal{A}_i \cap \mathcal{A}_{i,cont} \cap \mathcal{S}$ *is a non-degenerate interval and such that the econometrician has* ex ante *knowledge of* $\overline{x}_i(a_i', a_{-i})$ *and* $\overline{t}_i(a_i', a_{-i})$ *for all* $a_i' \in \mathcal{A}_i \cap \mathcal{A}_{i,cont} \cap \mathcal{S}$ *and* $a_{-i} \in \tilde{\mathcal{A}}_{-i}^d(a_i')$, *where* $\tilde{\mathcal{A}}_{-i}^d(a_i')$ *has probability* 1 *according to the distribution* $A_{-i}|(A_i = a_i)$. *The distribution* $A_{-i}|(A_i = a_i)$ *is point identified. If* $a_i \in int(\mathcal{A}_{i,cont})$, *then* $a_i \in int(\mathcal{A}_i \cap \mathcal{A}_{i,cont} \cap \mathcal{S})$. *The data is* $P(A)$, *or more.*

(5) [Game-structure identification with two-part transfers] *The transfer can be written as* $\widetilde{t}_i(a_i, a_{-i}) = \widetilde{t}_{i1}(a_i, a_{-i}) + \widetilde{t}_{i2}(a_i, a_{-i})$, *with corresponding expected transfers* $\overline{t}_i(a_i, a_{-i}) = \overline{t}_{i1}(a_i, a_{-i}) + \overline{t}_{i2}(a_i, a_{-i})$. *Correspondingly, the realized transfer* T_i *can be written as* $T_i = T_{i1} + T_{i2}$. *It holds that for all* a_i' *in some neighborhood* $\mathcal{N}$ *of* a_i *that* $\overline{t}_{i2}(a_i', a_{-i}) = \overline{t}_{i2}(a_{-i})$. *Any of Conditions 1-4 hold with* $\overline{t}_{i1}$ *in place of* $\overline{t}_i$ *and* T_{i1} *in place of* T_i.

Then, whether or not $\Psi_i^x(a_i)$ *and* $\Psi_i^t(a_i)$ *exist is point identified. Exists means, by definition, that the limit corresponding to the definition of the derivative exists. Moreover, if* $\Psi_i^x(a_i)$ *and* $\Psi_i^t(a_i)$ *exist, then there is game-structure identification per Definition 4. In the case of Condition 1 or 2, identification of* $\Psi_i^x(a_i)$ *and* $\Psi_i^t(a_i)$ *is constructive, and given by the existence and values of the limits corresponding to expressions in Equation 13. In the case of Condition 3, identification of* $\Lambda_i^x(a_i)$ *and* $\Lambda_i^t(a_i)$ *is constructive, and given by the existence and values of the limits corresponding to expressions in Equation 16. In the case of Condition 4, identification of* $\Psi_i^x(a_i)$ *and* $\Psi_i^t(a_i)$ *is constructive, and given by the existence and values of the limits corresponding to expressions in Equation 8.*

Condition 1 is based on the fact, in connection with standard results on identification and estimation of conditional expectations, that $\overline{x}_i(\cdot)$ and $\overline{t}_i(\cdot)$ are identifiable quantities based on the relationships $\overline{x}_i(a_i', a_{-i}) = E_P(X_i|A_i = a_i', A_{-i} = a_{-i})$ and $\overline{t}_i(a_i', a_{-i}) = E_P(T_i|A_i = a_i', A_{-i} = a_{-i})$. For example, kernel regression estimators of conditional expectations are consistent for almost all realizations of the conditioning variable, with respect to the distribution of the conditioning variable (e.g., Stone (1977), Devroye (1981), or Greblicki, Krzyzak, and Pawlak (1984)).[34] Condition 1 is also based

[33]By the standard formula for a conditional density, this would be implied by continuity of the joint density and marginal density, when θ_i^* is such that the marginal density is strictly positive. This can also hold more generally, for example it can hold even when θ_i^* is such that the marginal density is zero even though θ_i^* is in the support of the valuations (e.g., on the boundary), for example in particular under Assumption 2 (Independent valuations).

[34]Nevertheless, it is necessary to make the assumption that the conditional expectations are point identified for a_i' in the *interval* $\mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{S}$ in order to identify the derivative. There are pathological functions like Thomae's function defined on $[0, 1]$ that are continuous on a set of Lebesgue measure one (e.g., the irrationals in $[0, 1]$) but discontinuous on a set of Lebesgue measure zero yet dense set (e.g., the rationals in $[0, 1]$), and yet are nowhere differentiable, even though the limit corresponding to the definition of the derivative does exist along sequences restricted to the irrationals. So identifying the conditional expectations on a set of probability 1 is not quite enough to identify the derivatives, if the set of probability 0 where the conditional expectations are not identified is dense.

on standard results on identification and estimation of the conditional distribution $A_{-i}|(A_i = a_i)$. For example, kernel estimators of conditional distributions are consistent for almost all realizations of the conditioning variable, with respect to the distribution of the conditioning variable, and all realizations of the conditioning variable if $A_{-i}|(A_i = a_i)$ is suitably continuous in a_i (e.g., Stute (1986), Owen (1987), and Hall, Wolff, and Yao (1999)). Therefore, the most practically important part of Condition 1 relates to the support of the data. The support condition requires that $a_i \in \mathcal{A}_i^d$ (in addition to $a_i \in \mathcal{A}_{i,\text{cont}}$) and that there is a set $\mathcal{S}$ containing a_i such that $\mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{S}$ is a non-degenerate interval, with $a_i \in \text{int}(\mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{S})$ if $a_i \in \text{int}(\mathcal{A}_{i,\text{cont}})$. The support condition is used to identify the derivatives based on limits along a sequence of a_i' approaching a_i, where a_i' are taken in $\mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{S}$. The condition that $a_i \in \text{int}(\mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{S})$ if $a_i \in \text{int}(\mathcal{A}_{i,\text{cont}})$ is used to guarantee that the usual two-sided derivative can be identified (to exist), when a_i is such that the two-sided derivative is relevant.[35] The support condition can be checked in an application.

Moreover, Condition 2 provides a related sufficient condition. The set $(\text{int}(\mathcal{A}_{i,\text{cont}}) \cap \text{bd}(\mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}}))^C \cap (\mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}})$ is the set of all actions actually used from the continuous part of the action space, except for any action that is in the interior of the continuous part of the action space yet on the boundary of the set of actions actually used from the continuous part of the action space. In the common situation that $\mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}}$ is an interval, so the set of actions used from the continuous part of the action space form an interval, then $\text{bd}(\mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}})$ is the at most two actions on the boundary of this interval. Further, game-structure identification is only relevant for actions that are not used as mass points. Therefore, possibly not being able to achieve game-structure identification on this boundary corresponds (at most) to not being able to achieve game-structure identification for a probability zero set of actions. The condition that $\tilde{\tilde{\mathcal{A}}}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{I}$ is a non-degenerate interval means that there is a non-degenerate interval of actions used nearby the action a_i, none of which are used as a point mass in the distribution of A_i.[36] Condition 2 also assumes some continuity of the *ex post* expected allocation and *ex post* expected transfer, as a sufficient condition for point identification of the conditional expectations. For example, in the context of a contest from Example 1, continuity of the contest success function is sufficient. For another example, in the context of an auction from Example 2, the *ex post* expected allocation and *ex post* expected transfer of player i tend to be continuous when $a_i' \in \mathcal{A}_{i,\text{cont}}$ except for at a profile of bids that results in player i tied for

[35]Even though the ordinary two-sided derivative equals both one-sided derivatives, when the ordinary two-sided derivative exists, it is possible that a one-sided derivative exists despite the two-sided derivative not existing. Hence, the econometrician might be able to identify a one-sided derivative that does not equal the two-sided derivative, if the two-sided derivative does not exist. Of course, if the econometrician assumes that either one-sided derivative, if it exists, equals the usual two-sided derivative, then this distinction becomes irrelevant.

[36]As detailed in the proof, this condition is closely related to the condition that $a_i(\cdot)$ is continuous and strictly increasing at least on a small interval containing the valuation that uses the action a_i. Continuity on even a small neighborhood is not implied by strictly increasing since, although perhaps pathological as a property of a strategy, a strictly increasing function can have jump discontinuities at a countably dense subset of its domain (e.g., Rudin (1976, Remark 4.31)). Nevertheless, continuity and strictly increasing is a common property in games, at least on a small interval containing the valuation that uses such an action a_i (not as a mass point). See the references to the economic theory literature elsewhere in the paper. In particular, a common approach to characterizing the equilibrium strategy, when there is sufficient differentiability of the game, is via a differential equation involving the derivative of the strategy with respect to the valuation, which of course requires that the strategy is continuous. The identification analysis allows that there can be actions at which *ex interim* expected utility is not differentiable.

high bid with at least one other player. For any given bid of player i, the set of bids of the other players that results in a tie for high bid generally has probability zero in equilibrium, and therefore such discontinuities are accommodated by Condition 2 because continuity is required only on a set of probability 1. As a substitute for assuming point identification of $A_{-i}|(A_i = a_i)$, Condition 2 assumes that there is an interval $\mathcal{I}$ containing a_i such that there are no mass points in the distribution of A_i within that interval, and that $\theta_{-i}|\theta_i$ is suitably continuous in θ_i. Therefore, Conditions 1 and 2 apply under weak standard conditions.

Condition 3 is similarly based on standard identification of conditional expectations given the expressions in Equation 16.[37]

Condition 4 is the fact that knowledge of the allocation and transfer rules is another sufficient condition for game-structure identification.[38] Even when using Condition 4 to achieve game-structure identification, it is not necessary that the econometrician solves for the equilibrium of the game (or have *ex ante* knowledge of the selected equilibrium in cases of multiple equilibria). Knowledge of $\overline{x}_i(\cdot)$ and $\overline{t}_i(\cdot)$ concerns knowledge of the allocation and transfer rules, not the equilibrium of the game. Data on allocations and transfers is used in the identification strategy only to achieve game-structure identification. Consequently, if game-structure identification is satisfied entirely via Condition 4, then the identification strategy does not require data on allocations and transfers.

Condition 5 shows that game-structure identification can be achieved in games with "two-part" transfers even if only one "part" of the transfer can be game-structure identified. For example, in auctions with participation costs from Example 2, the total transfer $\widetilde{t}_i(\cdot)$ is the sum of the "standard" auction payment $\widetilde{t}_{i1}(\cdot)$ that accounts for who wins and loses the auction and the participation cost $\widetilde{t}_{i2}(\cdot)$ that depends only on the binary decision of participation in the auction (i.e., whether the player bids or takes the special "do not participate" action). The data on transfers and/or the econometrician's *ex ante* knowledge of the rule for transfers might correspond to only one part of the two-part transfer. For example, in auctions with participation costs, the econometrician might have data (or *ex ante* know) the "standard" payments to the auctioneer. However, particularly if the participation costs are not entirely imposed by the auctioneer (e.g., if the participation costs include the private costs of preparing a bid), then the econometrician might not have data (or *ex ante know*) the participation costs. Because only *marginal* transfers are relevant to game-structure identification, it is still possible to achieve game-structure identification. For example, in auctions with participation costs, for $a_i \geq 0$, the participation cost "part" of the transfer does not depend (in a sufficiently small neighborhood of such a_i) on a_i. Condition 5 applies the decomposition of the transfer only *locally* to a given action. Of course, the "participation cost part" of the transfer does depend on the action at the threshold between participating and not participating. But it does not depend on the action in a

[37]Per arguments similar to above, using Assumption 2, $E_P(X_i|A_i = a_i) = E_P(\overline{x}_i(a_i, A_{-i}))$ and $E_P(T_i|A_i = a_i) = E_P(\overline{t}_i(a_i, A_{-i}))$ tend to be continuous functions of a_i under weak conditions on the game. Or see Footnote 17.

[38]When using Condition 4 to achieve game-structure identification, the econometrician is allowed to not have *ex ante* knowledge of $\overline{x}_i(a_i, a_{-i})$ and $\overline{t}_i(a_i, a_{-i})$ for all $a = (a_1, a_2, \ldots, a_N)$. For example, in an application to auction models, for given a_i', the econometrician need not know the tie-breaking rule that determines $\overline{x}_i(a_i', a_{-i})$ and $\overline{t}_i(a_i', a_{-i})$ in the case that a_{-i} is such that there is a tie for high bid, supposing that the conditional probability of such ties is zero with respect to the distribution of the data. Of course, in many auction formats the econometrician might have the *ex ante* knowledge that the tie-breaking rule allocates the object amongst the tied bidders with equal probability.

sufficiently small neighborhood of any participating bid. Hence, the *marginal* expected transfer does not depend on the participation cost for participating bids. In such cases, Condition 5 shows that it is possible to achieve game-structure identification by applying Conditions 1-4 to the $t_{i1}(\cdot)$ "part" of the transfer. For example, from Conditions 1-3, it is possible to achieve game-structure identification based on observing the "standard" payments to the auctioneer, but not the participation cost. And from Condition 4, it is possible to achieve game-structure identification based on *ex ante* knowledge of the rule for the "standard" payments to the auctioneer, but not the participation cost.

Appendix B. Examples of games

The class of incomplete information games studied in this paper is illustrated via examples.

Example 1 (Contests)**.** This incomplete information game framework includes contest models, in which the actions are interpreted as "costly effort" toward winning a valuable object. The economic theory of such models has been developed in, for example, Hillman and Riley (1989), Baye, Kovenock, and De Vries (1993), Amann and Leininger (1996), Krishna and Morgan (1997), Lizzeri and Persico (2000), and Parreiras and Rubinchik (2010), in addition to an overall large literature. See for example Konrad (2007, 2009) for a summary of the literature, including discussion of theoretical applications to a broad range of instances of competition, including advertising, litigation, political lobbying, research and development, and sports.[39]

The valuation θ_i is the value that player i has for the object. Often, the "efforts" are equivalent to financial expenditures, so that $\mathcal{A}_i = [0, \infty)$ and the transfer rule is $\bar{t}_i(a) = a_i$. However, other transfer rules are also possible. For example, it might be that part of the effort is "refundable," so that players only expend some fraction of their effort, possibly depending on whether the player wins or loses (e.g., see the models in Riley and Samuelson (1981) and Matros and Armanios (2009)). The allocation rule $\overline{x}(a) = (\overline{x}_1(a), \overline{x}_2(a), \ldots, \overline{x}_N(a))$ is known as the "*contest success function*" that relates the actions taken by the players to the probabilities that each of the players wins the valuable object. The econometrician may not know the contest success functions $\overline{x}(\cdot)$, and indeed the economic theory literature has explored a variety of different contest success functions. See for example Corchón and Dahm (2010) for a detailed discussion. For example, following Tullock (1980)-style models, $\overline{x}_i(a) = \begin{cases} \frac{a_i^r}{\sum_{j=1}^N a_j^r} & \text{if } a \neq 0 \\ \frac{1}{N} & \text{if } a = 0 \end{cases}$ for some $r > 0$. In particular, the case of $r = 1$ has been interpreted as a "lottery" in which the probability that player i wins is equal to player i's share of the overall aggregate effort. The specification states that if all players expend no effort, then each player has equal chance of winning the contest. More generally, there can be functions $f_i(\cdot)$ such that $\overline{x}_i(a) = \frac{f_i(a_i)}{\sum_{j=1}^N f_j(a_j)}$, including the logistic specification $f_i(z) = e^{kz}$ as in Hirshleifer (1989). Alternatively, following Lazear and Rosen (1981)- and Dixit (1987)-style models, $\overline{x}_i(a) = P_\epsilon(a_i + \epsilon_i > \max_{j \neq i}(a_j + \epsilon_j))$, where P_ϵ is the distribution of "noise" or "randomness" involved in determining the contest winner. Because the identification results do not require a complete specification of the game, the identification results do

[39] These models fit the frameworks of the papers establishing conditions for monotone equilibrium in games (discussed further after Assumption 6), as illustrated for example by Wasser (2013) who applies Athey (2001) to establish conditions for a monotone equilibrium in contests.

not require the econometrician to know $\overline{x}(\cdot)$ (or the underlying distribution $\widetilde{x}(\cdot)$). In particular, the econometrician might not know r or f_i or P_ϵ.

In the above specifications, generally a player that expends the most effort is most likely to win, but is not guaranteed to win. In the limiting case of the "all-pay auction" formulation,

$$\overline{x}_i(a) = \begin{cases} 1 & \text{if } i \text{ expends the most effort} \\ p_i(a) & \text{if } i \text{ ties for expending the most effort with at least one other player} \\ 0 & \text{if } i \text{ does not expend the most effort,} \end{cases}$$

where $p_i(a)$ reflects the tie-breaking rule. In all-pay auction models, the player that expends the most effort is guaranteed to win.

Example 2 (Auctions)**.** This incomplete information game framework includes auction models, including auction formats involving various complications like "participation costs," reserve prices, asymmetries, and/or multiple units possibly with endogenous supply. The economic theory of auctions is too large to attempt to even partly review here, but has been reviewed, for example, in Klemperer (1999, 2004), Milgrom (2004), and Krishna (2009). One feature of the auction theory literature is the range of auction formats, implying a range of allocation and transfer rules.[40] The identification strategy can apply to a wide range of auction formats, because the identification strategy applies to this class of incomplete information games, which includes a wide variety of auction formats. One of many possible auction models fitting the framework is discussed here.[41]

The valuation θ_i is the value player i has for a unit of the object being auctioned. Because this incomplete information game framework does not necessarily require the assumption of symmetric players, the auction could involve such asymmetries as "strong" and "weak" bidders, as in Milgrom (2004, Section 4.5).[42] The action space is $\mathcal{A}_i = \{DNP\} \cup [r_i, \infty)$, where as discussed above, the "DNP" action has a special interpretation as "do not participate in the auction" and $r_i \geq 0$ is the reserve price. The transfers include the payments to the auctioneer, but could include participation costs when applicable. The allocation is the awarding of units of the object from the auction. The allocation rule and transfer rule depend on the specifics of the auction format. A participation cost can be modeled in a few different ways, particularly concerning whether or not the players know their own valuation at the time they make the participation decision.[43] This example concerns the

[40]The economic theory of auctions with participation costs has been developed in, for example, Samuelson (1985), McAfee and McMillan (1987), Levin and Smith (1994), Tan and Yilankaya (2006), and Cao and Tian (2010). See for example (Krishna, 2009, Section 2.5) for equilibrium in auctions with reserve prices.

[41]Much of the economic theory literature has focused on establishing monotonicity of the strategy in auction models, and moreover the literature on general conditions for monotone equilibrium in games (discussed further after Assumption 6) often treats auctions as a leading example of their results.

[42]For example, Campo, Perrigne, and Vuong (2003) have focused on establishing point identifying assumptions for asymmetric bidders with affiliated private values in first price auctions. Reny and Zamir (2004) have studied the existence of monotone equilibrium in related auction models.

[43]A third approach allows that bidders observe a signal of their valuation at the time of their participation decision, an identification problem studied in Gentry and Li (2014). Other identification results emphasizing entry/participation in particular auction models includes Marmer, Shneyerov, and Xu (2013) (focusing on identifying the selection effect, and discriminating between models of entry), Fang and Tang (2014) (focusing on inferring bidder risk attitudes), and Li, Lu, and Zhao (2015) (focusing on testable implications of risk aversion). In this paper, the identification problem concerns the partial identification of the distribution of valuations.

case that bidders know their own valuation at the time they make the participation decision (e.g., Samuelson (1985), Tan and Yilankaya (2006), and Cao and Tian (2010)).

Let $r_i \geq 0$ be the reserve price for player i. Generally, with symmetric players, $r_i = r = r_j$, but with asymmetric players, reserve prices could be player-specific. Suppose that there is endogenous supply, in the sense that the quantity allocated to the winning bidder is a function $S(a)$ of the profile of bids (e.g., Milgrom (2004, Section 4.3.3)). For example, the supply $S(a)$ might depend only on the winning bid, as in a "supply curve" at the "price" of the winning bid. See also Example 3 for related models where $S(a)$ can be interpreted as a "demand curve." The standard case that there is one exogenous unit of the object being auctioned is the special case that $S(\cdot) \equiv 1$. Let $H_i(a) = \max_{j \neq i \text{ and } j \text{ s.t. } a_j \geq r_j} a_j$ be the highest bid other than the bid of player i, among the bids from players that exceed the corresponding reserve price.

Then, in auction formats where the highest bidder wins, as long it exceeds its reserve price and the highest competitor's bid among those bids exceeding the corresponding reserve price,[44]

$$\overline{x}_i(a) = \begin{cases} S(a) & \text{if } a_i > H_i(a) \text{ and } a_i \geq r_i \\ p_i(a) & \text{if } a_i = H_i(a) \text{ and } a_i \geq r_i \\ 0 & a_i < H_i(a) \text{ or } a_i < r_i, \end{cases}$$

where $p_i(a) \in [0, S(a)]$ reflects the tie-breaking rule, the expected number of units that player i is allocated when bids are a, involving a tie for high bid. The transfer rule depends on the auction format. But in many auction formats including those with participation costs, the transfer rule can be written $\tilde{t}_i(a) = \tilde{t}_{i1}(a) + \tilde{t}_{i2}(a)$, where $\tilde{t}_{i1}(\cdot)$ is the auction payment rule that accounts for who wins and loses the auction, and $\tilde{t}_{i2}(\cdot)$ is the participation cost that depends only on the binary decision of participation in the auction (i.e., whether the player bids or takes the special "do not participate" action). Hence, with participation cost c,

$$\bar{t}_{i2}(a) = \begin{cases} c & \text{if } i \text{ participates (i.e., } a_i \geq 0) \\ 0 & \text{if } i \text{ does not participate (i.e., } a_i = DNP) \end{cases}$$

Then, for example in a first price auction, and noting that $\bar{t}_{i1}(a)$ is the *expected* transfer that integrates over the tie-breaking rule,

$$\bar{t}_{i1}(a) = \begin{cases} a_i S(a) & \text{if } a_i > H_i(a) \text{ and } a_i \geq r_i \\ a_i p_i(a) & \text{if } a_i = H_i(a) \text{ and } a_i \geq r_i \\ 0 & a_i < H_i(a) \text{ or } a_i < r_i \end{cases}$$

Other auction formats would have different allocation rules and/or transfer rules.

Some participation costs may be paid directly to the auctioneer, while other participation costs are not paid to the auctioneer. The participation costs could include unobserved costs like the "cost of preparing a bid" or the "opportunity cost of participating in the auction." Consequently, the econometrician may not know $\bar{t}_i(a)$ (or the underlying distribution $\tilde{t}_i(a)$), because the econometrician

[44]The comparison of actions to the reserve price guarantees that a player that takes an action below the reserve price (particularly DNP) does not win the auction.

may not know c appearing in $\bar{t}_{i2}(a)$, but because the identification results do not require a complete specification of the game, the identification results do not require the econometrician to know $\bar{t}_i(a)$. In particular, the participation cost need not be observed or known by the econometrician. See the discussion of Condition 5 of Lemma 1. Similarly, the econometrician may not know $\overline{x}_i(a)$ and/or $\bar{t}_i(a)$, because the econometrician may not know the "supply function" $S(a)$, but again, the identification results do not require the econometrician to know $\overline{x}_i(a)$ and/or $\bar{t}_i(a)$.

Intuitively, players with low valuations refuse to participate in equilibrium. See for example Menezes and Monteiro (2005, Section 3.1.4). Therefore, the partial identification strategy can intuitively be expected to result in an upper bound on the valuations corresponding to players that do not participate, and point identification of valuations corresponding to players that do participate.[45] Because the identification strategy does not restrict to a particular model of the auction, the identification result will vary depending on the identifying content of the data based on the specifics of the auction.

Of course, the identification problems presented by specific auction models have been treated in isolation. The point of this example is to further show the generality of this incomplete information game framework, where such models are examples of a broader identification strategy (with the features outlined in the introduction and the rest of the paper), even relative to auction models specifically. For example, as cited above, there have been papers specifically focusing on the identification problem posed by a participation cost, and other papers focusing on the identification problem posed by asymmetric bidders, and so forth. In contrast, this general incomplete information game framework flexibly accommodates various combinations of such complications in auctions without the need for a "specialized" identification strategy, alongside perhaps even other complications. And, of course, the framework extends beyond auctions to other settings. In some cases, for example as in some of the existing literature on participation costs cited in Footnote 43, the identification problems addressed have concerned objects of interest other than the underlying distribution of valuations, and establishing those objects are point identified, or have testable implications, and so forth. This paper focuses always on the distribution of valuations, even if that happens to be partially identified.

Example 3 (Procurement auctions, reverse auctions, oligopoly models, etc.)**.** Models of procurement auctions, reverse auctions, and related situations fit this incomplete information game framework. Such models are similar to auctions, with the distinguishing feature that the N players are bidding to *sell* units of an object, rather than *buy* units of an object. Therefore, the valuation θ_i can be interpreted to be player i's (constant) marginal cost of supplying one unit of the object, and the "low bid" wins the market. Let $L_i(a) = \min_{j \neq i \text{ and } j \text{ s.t. } a_j \leq r_j} a_j$ be the lowest bid other than the bid of player i, among the bids from players that are below the corresponding reserve price. The "allocation" experienced by player i is the quantity of the object that player i supplies, and therefore the allocation

[45] If players do not know their own valuation when they make the participation decision (e.g., McAfee and McMillan (1987) and Levin and Smith (1994)), intuitively the consequence is that players use a mixed strategy to determine their participation with the result that the eventual auction is essentially an auction with fewer players than would otherwise be in the auction, to compensate (in equilibrium) the participating players for paying the participation cost, but with no relation between a players's participation and valuation. See Milgrom (2004, Section 6.2).

is negative, so the allocation rule could be

$$\overline{x}_i(a) = \begin{cases} -S(a) & \text{if } a_i < L_i(a) \text{ and } a_i \leq r_i \\ -p_i(a) & \text{if } a_i = L_i(a) \text{ and } a_i \leq r_i \ , \\ 0 & a_i > L_i(a) \text{ or } a_i > r_i, \end{cases}$$

where, similarly to Example 2, $S(a)$ is the endogenous quantity (i.e., "demand") given the profile of bids a, r_i is the maximum acceptable bid for player i, and $p_i(a)$ reflects the tie-breaking rule. The "transfer" experienced by player i is the payment to player i. Due to the convention in this paper that transfers are *from* the player, transfers are negative. For example, it could be that

$$\overline{t}_i(a) = \begin{cases} -a_i S(a) & \text{if } a_i < L_i(a) \text{ and } a_i \leq r_i \\ -a_i p_i(a) & \text{if } a_i = L_i(a) \text{ and } a_i \leq r_i \\ 0 & a_i > L_i(a) \text{ or } a_i > r_i \end{cases}$$

Some models of oligopoly competition are basically the same game, with N firms in an oligopoly having privately known constant marginal costs of production competing to win the oligopoly market, see for example Vives (2001, Chapter 8). In these models, the "endogenous quantity" $S(a)$ is the demand curve, generally depending on the lowest bid (i.e., the "realized price"). As with the endogenous supply in Example 2, the econometrician may not know the "demand curve" and therefore again not know $\overline{x}_i(a)$ and/or $\overline{t}_i(a)$, but again the identification results do not require the econometrician to know $\overline{x}_i(a)$ and/or $\overline{t}_i(a)$.

Example 4 (Partnership dissolution)**.** Models of partnership dissolution and related situations fit the framework. The economic theory of such models has been developed in Cramton, Gibbons, and Klemperer (1987), in addition to a huge subsequent literature. There are N co-owners of an object. Prior to partnership dissolution, player i owns share r_i of the object and has valuation θ_i for the object. The econometrician need not know these ownership shares.

In the "bidding game" formulation of partnership dissolution developed in Cramton, Gibbons, and Klemperer (1987), there are initial transfers between the co-owners that depend on their ownership shares. Since these initial transfers do not depend on valuations, they are not revealing of valuations. In the special case of equal ownership shares, these initial transfers are zero. In any case, the econometrician need not observe data on these initial transfers in order to apply the identification strategy. Indeed, the identification strategy does not rely on the game implementing such initial transfers. (These initial transfers are for purposes of satisfying the individual rationality constraint, violation of which does not change the identification strategy in this paper, since this paper essentially only uses the incentive compatibility constraint. See formula C of Cramton, Gibbons, and Klemperer (1987, Theorem 2).) Then, each co-owner bids for ownership, so the action in the game are bids, with the highest bidder winning ownership. The transfer from player i is (omitting the "lump sum" initial transfer reflecting ownership shares but not valuations): $\overline{t}_i(a) = a_i - \frac{1}{N-1}\sum_{j \neq i}^{N} a_j$, so player i transfer its bid even if it loses, and is "compensated" by the bids of the other players.[46]

[46]Cramton, Gibbons, and Klemperer (1987) works under the assumption of independent valuations. Per the proof of Cramton, Gibbons, and Klemperer (1987, Theorem 2), the equilibrium is found in strictly increasing strategies.

Example 5 (Public good provision)**.** Models of the provision of public goods or public projects, and related situations, fit the framework. The distinguishing feature of such models is that the allocation is the same to all players, reflecting the "public" nature of the object. The valuation θ_i reflects the private value that player i places on the public good. The economic theory of such models has been developed in Bergstrom, Blume, and Varian (1986), Bagnoli and Lipman (1989), Mailath and Postlewaite (1990), Alboth, Lerner, and Shalev (2001), Menezes, Monteiro, and Temimi (2001), and Laussel and Palfrey (2003), in addition to a huge overall literature, summarized for example in Ledyard (2006).[47] The games studied for public good provision differ significantly, and so a complete discussion is not feasible here. In direct revelation games (e.g., Clarke (1971)-Groves (1973) games), players report their valuation, in which case the identification problem is trivial. In other games, the actions of the players are interpreted as contributions to the public good, and the object is allocated (e.g., the public project is completed) if and only if the sum of the contributions of the players is greater than the cost of producing the public good. The contributions may or may not be refunded if the public good is not produced, depending on the specific game. See for example Menezes, Monteiro, and Temimi (2001). Some models of public good provision, along the lines of Palfrey and Rosenthal (1984) (who worked with complete information), involve a discrete and even binary action space (contribute an *ex ante* fixed amount or not), providing motivation for identification results with discrete action spaces in Appendix D.

Appendix C. On the role of equilibrium assumptions

Bayes Nash equilibrium requires that all players act rationally given beliefs (Assumption 4) and have correct beliefs (Assumption 5), so that each player chooses an action that is a best response to the distribution of actions of the other players. It is relevant to ask what role equilibrium played in the identification results, both to gain a better understanding of the identification results, and also because it may be useful in some settings to relax the assumption of equilibrium.

This assumption of equilibrium is completely standard, since it is reasonable in very many settings, but in some settings it may be too strong.[48] In the context of auction models, for example, it might be that some "novice" bidders have incorrect beliefs about the other bidders, whereas "experienced" bidders might have correct beliefs about the other bidders. Similarly, it might be that the "novice" bidders do not have sufficient understanding or experience with the auction format to bid the optimal amount given their beliefs, whereas "experienced" bidders might have that sufficient understanding and experience to bid the optimal amount given their beliefs. The difference between "novice" and "experienced" might be due to learning from participating in previous auctions, or some other reason that is observable by the econometrician, so that the econometrician can distinguish which players are "novices" and which players are "experienced." For example, in electricity markets with data that

[47] See Lemma 1 or the discussion of "regular" equilibrium in Laussel and Palfrey (2003) for the role of monotonicity in the strategies. Or see the characterization of the equilibrium strategies in Menezes, Monteiro, and Temimi (2001).

[48] Identification in games relaxing the assumption of equilibrium, or related questions, has been considered in Aradillas-Lopez and Tamer (2008), Haile, Hortaçsu, and Kosenok (2008), Kline and Tamer (2012), and Kline (2015, 2016b). Kline (2016a) includes a discussion of the tradeoffs between equilibrium assumptions and assumptions on the data, for identification in settings like entry games. See Maskin (2011) for a commentary on Nash equilibrium.

includes typically unobserved valuations, which makes it possible to test bidder optimality, Hortacsu and Puller (2008) find that "large" firms are more strategically sophisticated than "small" firms.

It is possible to identify valuations for any player that has correct beliefs and acts rationally given those correct beliefs, regardless of whether other players have correct beliefs and/or act rationally given those beliefs. Therefore, it is possible to identify particular players' valuations, even without the assumption of equilibrium. Moreover, such a result is useful to understand the role of equilibrium assumptions in Theorems 1 and 2. As before, the identification strategy involves recovering the valuation corresponding to an action, based on the *ex interim* utility maximization problem from Equation 1. Under the full assumption of equilibrium, this resulted in identification of the valuations of all players. Under the weaker assumptions in this section, this results in identification of the valuation of player i. In other words, it is possible to identify the valuation of player i, without assuming anything about the behavior of the other players, as long as player i still has correct beliefs and acts rationally given those correct beliefs. Those beliefs need not involve the other players themselves having correct beliefs or acting rationally given those beliefs. For example, player i might have correct beliefs that the other players are "irrational."

If it were assumed that all players draw valuations from the same marginal distribution (i.e., "*symmetric private values*"), then identification of one player's marginal distribution of valuations is sufficient to identify all players' marginal distributions of valuations. If it were further assumed that player valuations are independent (i.e., "*symmetric independent private values*"), then identification of one player's marginal distribution of valuations is sufficient to identify the *joint distribution* of all players' distributions of valuations. Of course, it may be implausible to assume that only some players have correct beliefs and act rationally given those beliefs, while also assuming that all players draw valuations from the same marginal distribution. However, if for example all players have the same marginal distribution of valuations, but some players just happen to have more "experience" with the game for reasons unrelated to their valuation, those simultaneous assumptions may be plausible. Of course, in any case, it is possible to identify the valuations of the player that has correct beliefs and acts rationally given those correct beliefs.

The identification strategy is almost exactly the same as developed in Sections 3 and 4. When relaxing the assumption of equilibrium, only a specific player i is assumed to have correct beliefs and act rationally given those beliefs. Assumptions 4, 5, 6, 9, 10, 11, and 12 that apply to all players are replaced by similar assumptions that apply only to a particular player i:

Assumption 4i (Player i uses an optimal strategy)**.** For each possible valuation θ_i, player i uses a strategy $a_i(\theta_i)$ when it has valuation θ_i, with $a_i(\theta_i) \in \Delta(\arg\max_{a_i \in \mathcal{A}_i} V_i(\theta_i, a_i))$, so each action taken according to the strategy $a_i(\theta_i)$ maximizes *ex interim* expected utility.

Assumption 5i (Player i has correct beliefs)**.** Player i has correct beliefs, in the sense that, for each possible valuation θ_i, $\Pi_i(a_{-i} \in B|\theta_i) = P(A_{-i} \in B|\theta_i)$ for all Borel sets B.

Assumption 6i (Player i uses a weakly increasing strategy)**.** For each possible valuation θ_i, $a_i(\theta_i)$ is a pure strategy. And, $a_i(\cdot)$ is a weakly increasing function.

Assumption 9i (Player i has no use of discrete actions and no point masses in distribution of actions)**.** $\mathcal{A}_i^d \subseteq \mathcal{A}_{i,\text{cont}}$ and $\tilde{\tilde{\mathcal{A}}}_i^d = \mathcal{A}_i^d$.

Assumption 10i (Player i has differentiable *ex interim* expected allocation and transfer)**.** There is a set $\mathcal{E}_{i,d}$ with $P(A_i \in \mathcal{E}_{i,d}) = 0$ such that the game and player i's beliefs are such that, for each possible valuation θ_i, the expected allocation $E_{\Pi_i}(\overline{x}_i(a_i, a_{-i})|\theta_i)$ and the expected transfer $E_{\Pi_i}(\overline{t}_i(a_i, a_{-i})|\theta_i)$ are differentiable functions of a_i, evaluated at any $a_i^* \in \text{support}(a_i(\theta_i)) \cap \mathcal{E}_{i,d}^C$.

Assumption 11i (Player i has game-structure identification)**.** There is a set $\mathcal{E}_{i,r}$ with $P(A_i \in \mathcal{E}_{i,r}) = 0$ such that if $a_i \in \mathcal{A}_i^d \cap \mathcal{E}_{i,r}^C$ is such that $\Psi_i^x(a_i)$ and $\Psi_i^t(a_i)$ exist then a_i is an action with game-structure identification per Definition 4.

Assumption 12i (Player i experiences non-zero marginal expected allocation)**.** There is a set $\mathcal{E}_{i,m}$ with $P(A_i \in \mathcal{E}_{i,m}) = 0$ such that $\Psi_i^x(a_i) \neq 0$ for $a_i \in \mathcal{A}_i^d \cap \mathcal{E}_{i,m}^C$.

Theorem 3. *Under Assumptions 1 (Dependent valuations), 3 (Action space), 4i (Player i uses an optimal strategy), 5i (Player i has correct beliefs), 6i (Player i uses a weakly increasing strategy), 9i (Player i has no use of discrete actions and no point masses in distribution of actions), 10i (Player i has differentiable* ex interim *expected allocation and transfer), 11i (Player i has game-structure identification), and 12i (Player i experiences non-zero marginal expected allocation), the distribution of valuations θ_i is point identified, and the identification is constructive, because the distribution of θ_i equals the distribution of $\Psi_i(A_i)$, where A_i is distributed according to the data $P(A, X, T)$ and $\Psi_i(\cdot)$ is the identifiable function given in Equation 9 (see Lemma 1).*

Independent valuations *With independent valuations: replace Assumption 1 (Dependent valuations) with Assumption 2 (Independent valuations), drop Assumption 6i (Player i uses a weakly increasing strategy) and replace the Ψ functions with the Λ functions defined in Equation 15.* ★

Essentially, Theorem 3 is the "player i part" of Theorem 2, both in terms of assumptions and result. Hence, perhaps surprisingly, it is possible to *point identify* the distribution of valuations of player i without the assumption of equilibrium. For example, it could be that player i in the game is the "large/experienced" firm, in which case the assumptions would be, roughly, that the "large/experienced" firm acts rationally given beliefs and has correct beliefs, and the result would be that the distribution of valuations for the "large/experienced" firm would be point identified. Analogously, it is possible to formulate the "player i part" of Theorem 1, establishing partial identification of player i's distribution of valuations. Moreover, assuming that players i and j both satisfy the assumptions, it is possible to formulate the "player i and j part" of the identification results, establishing identification of the *joint* distribution of their valuations. In the interest of space, those results are not stated here.

Appendix D. Identification with an additional assumption

It is possible to extend the identification strategy under an additional assumption, which is especially useful for games involving discrete action spaces and/or non-differentiable *ex interim* expected allocation or *ex interim* expected transfer. The main identification results from Sections 3 and 4 deliver informative partial identification (and point identification under the conditions of Section

4) in the large class of incomplete information games that have at least partly continuous action spaces and somewhere differentiable *ex interim* expected allocation and *ex interim* expected transfer, corresponding to the large economic theory literature concerning games with these properties. However, in an extreme case, if the action space is entirely discrete, or if *ex interim* expected allocation and/or *ex interim* expected transfer are nowhere differentiable, then the identification result in Theorem 1 still applies, but as discussed after the statement of Theorem 1, the resulting identified set for the distribution of valuations would be the trivial bounds that the valuations are between the *ex ante* known bounds on the valuations from Assumption 8 (Known bounds on valuations). The identification strategy developed in this section establishes informative non-trivial bounds on the valuations even in that extreme case.

Indeed, in some games, the action space does not contain a "continuous part," which also implies that *ex interim* expected allocation and *ex interim* expected transfer cannot be a differentiable function of the action. For example, some auction formats might allow only bids that are an integer multiple of some fixed amount (e.g., the allowed bids might be 5 dollars, 10 dollars, 15 dollars, etc.).[49][50] Discrete action spaces could also arise in games other than auctions, as in Example 5.

In a sense formalized below, the identification strategy developed in this section is a sort of "discrete analogue" of the identification strategy developed in Section 3. An action from $\mathcal{A}_{i,\text{disc}}$ is generically used by multiple valuations. Consequently, an action from $\mathcal{A}_{i,\text{disc}}$ generically results in partial identification of the corresponding valuation. For the same reason, it is generically not possible to recover the beliefs of a player using an action from $\mathcal{A}_{i,\text{disc}}$, as done in Section 3. Therefore, a different approach to essentially *bounding* the beliefs of the players must be taken for actions taken from $\mathcal{A}_{i,\text{disc}}$.

Under Assumption 4 (Optimal strategy), for any valuation θ_i, any action $\tilde{a}_i(\theta_i)$ that solves the utility maximization problem in Equation 1 satisfies

$$\theta_i E_{\Pi_i}(\overline{x}_i(\tilde{a}_i(\theta_i), a_{-i})|\theta_i) - E_{\Pi_i}(\overline{t}_i(\tilde{a}_i(\theta_i), a_{-i})|\theta_i) \geq \tag{21}$$

[49]The economic theory of such auctions has been developed in Chwe (1989), Rothkopf and Harstad (1994), Dekel and Wolinsky (2003), David, Rogers, Jennings, Schiff, Kraus, and Rothkopf (2007). Also, some results on equilibrium existence including Milgrom and Weber (1985) and Athey (2001) use a finite action space as a theoretical construction.

[50]"Discrete" can be used with different definitions, which are worth distinguishing. Hortaçsu and McAdams (2010) studies an identification problem (and empirical application) in discriminatory price divisible goods auctions with independent private values. Kastl (2011) studies an identification problem (and empirical application) in uniform price divisible good auctions with (mainly) independent private values. In those models, bidders submit a bid function that specifies a quantity demanded for each possible price. Hence, neither model is covered by the incomplete information game framework studied in this paper, because those models deal with an action space that is a bid function rather than just a scalar bid. More importantly, the notion of "discrete" action is also different. In particular, Kastl (2011) uses "discrete" (per Kastl (2011, Assumption 3)) as a statement about the step function nature of the bid functions, where each player submits a bid function that is a step function, and therefore characterizable by a discrete vector of prices and quantities that characterize each "step" of the bid function. Hortaçsu and McAdams (2010) similarly emphasize step bid functions. However, the actual price and quantities at each step of the bid function is unrestricted. By contrast, as applied to auctions, this paper uses discrete as a statement on the restriction of the allowed bid levels. So, the players can only bid, for example, integer multiples of some minimal bid level. An earlier version of Hortaçsu (2002) looked at a model with a discrete grid of possible prices, and hence with a "discrete" action space more similar to the discreteness in this paper. Of course, the overall identification problem (and hence identification strategy) is still different from the identification problem addressed in this paper, particularly given the differences in the models being identified. The identification strategy in this paper does not restrict to auctions or independent values.

$$\max_{z_i \in \mathcal{A}_i} \left(\theta_i E_{\Pi_i}(\overline{x}_i(z_i, a_{-i})|\theta_i) - E_{\Pi_i}(\overline{t}_i(z_i, a_{-i})|\theta_i)\right).$$

Under Assumption 5 (Correct beliefs), Equation 21 implies

$$\theta_i E_P(\overline{x}_i(\tilde{a}_i(\theta_i), A_{-i})|\theta_i) - E_P(\overline{t}_i(\tilde{a}_i(\theta_i), A_{-i})|\theta_i) \geq \max_{z_i \in \mathcal{A}_i} \left(\theta_i E_P(\overline{x}_i(z_i, A_{-i})|\theta_i) - E_P(\overline{t}_i(z_i, A_{-i})|\theta_i)\right). \tag{22}$$

Under Assumption 6 (Weakly increasing strategy), for any action $a_i^* \in \mathcal{A}_i$ there is an interval

$$\Theta_i(a_i^*) = \{\theta_i : a_i(\theta_i) = a_i^*\} \tag{23}$$

of valuations such that player i with valuation θ_i uses action a_i^* if and only if $\theta_i \in \Theta_i(a_i^*)$. Moreover, if $a_i \neq a_i'$ then $\Theta_i(a_i)$ and $\Theta_i(a_i')$ are disjoint; and if $a_i < a_i'$ and $\Theta_i(a_i)$ and $\Theta_i(a_i')$ are both non-empty then $\sup \Theta_i(a_i) \leq \inf \Theta_i(a_i')$. Therefore, for any $z_i, z_i' \in \mathcal{A}_i^d$,

$$E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i') = E_P(\overline{x}_i(z_i, A_{-i})|\theta_i \in \Theta_i(z_i')) = E_P(E_P(\overline{x}_i(z_i, A_{-i})|\theta_i)|\theta_i \in \Theta_i(z_i')) \tag{24}$$

$$E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i') = E_P(\overline{t}_i(z_i, A_{-i})|\theta_i \in \Theta_i(z_i')) = E_P(E_P(\overline{t}_i(z_i, A_{-i})|\theta_i)|\theta_i \in \Theta_i(z_i')). \tag{25}$$

Hence, if θ_i corresponds to the use of $z_i' \in \mathcal{A}_{i,\text{disc}}$, the beliefs expressions in Equation 22 conditioning on θ_i are generically not identifiable by the same strategy as in Sections 3 and 4, because generically multiple valuations use any given $z_i' \in \mathcal{A}_{i,\text{disc}}$. However, it is possible to provide bounds on these beliefs expressions under an additional assumption. In order to state the additional assumption, as a regularity condition it is necessary to assume that *ex interim* expected utility has a maximizer when player i has valuation θ_i and "counterfactually" has the beliefs of valuation θ_i''.

Assumption 13 (Counterfactual *ex interim* expected utility maximization problem has a solution)**.** *For each* $i \in \{1, 2, \ldots, N\}$, $\max_{a_i \in \mathcal{A}_i} \left(\theta_i E_{\Pi_i}(\overline{x}_i(a_i, a_{-i})|\theta_i'') - E_{\Pi_i}(\overline{t}_i(a_i, a_{-i})|\theta_i'')\right)$ *has a solution for any possible valuations* θ_i *and* θ_i''.

Assumption 4 (Optimal strategy) states that the *ex interim* expected utility maximization problem has a solution when $\theta_i = \theta_i''$. Standard conditions imply that a solution exists even when $\theta_i \neq \theta_i''$.

Assumption 14 (Monotone effect of counterfactual beliefs on utility)**.** *For each* $i \in \{1, 2, \ldots, N\}$, *and any possible valuations* θ_i *and* θ_i'', *there is some selection*

$$a_i(\theta_i; \theta_i'') \in \arg\max_{a_i \in \mathcal{A}_i} \left(\theta_i E_{\Pi_i}(\overline{x}_i(a_i, a_{-i})|\theta_i'') - E_{\Pi_i}(\overline{t}_i(a_i, a_{-i})|\theta_i'')\right)$$

with $a_i(\theta_i; \theta_i) = a_i(\theta_i)$ *from Assumption 6, such that for* $a_i^* = a_i(\theta_i; \theta_i'')$ *and when* $\theta_i' \leq \theta_i''$,

$$\theta_i E_{\Pi_i}(\overline{x}_i(a_i^*, a_{-i})|\theta_i') - E_{\Pi_i}(\overline{t}_i(a_i^*, a_{-i})|\theta_i') \geq \theta_i E_{\Pi_i}(\overline{x}_i(a_i^*, a_{-i})|\theta_i'') - E_{\Pi_i}(\overline{t}_i(a_i^*, a_{-i})|\theta_i'').$$

Independent valuations *Under Assumption 2 (Independent valuations), Assumptions 13 and 14 are not used.* ★

The action $a_i^* = a_i(\theta_i; \theta_i'')$ maximizes the "counterfactual" *ex interim* expected utility of player i with valuation θ_i and "counterfactually" the beliefs of a player with valuation θ_i'' possibly not equal to θ_i. The assumption states that the "counterfactual" *ex interim* expected utility experienced by player i that has valuation θ_i that uses such an action a_i^* and "counterfactually" has the beliefs of valuation θ_i' with $\theta_i' \leq \theta_i''$ is weakly greater than that under the beliefs with valuation θ_i''. A sufficient condition is that $\theta_i E_{\Pi_i}(\overline{x}_i(a_i^*, a_{-i})|\theta_i') - E_{\Pi_i}(\overline{t}_i(a_i^*, a_{-i})|\theta_i')$ is a weakly decreasing function of θ_i'. Hence, the assumption can be interpreted as stating that utility is monotone in the "counterfactual beliefs" arising due to "counterfactual" valuations.

If valuations are independent, then beliefs do not depend on valuation, so this assumption is satisfied. Further, even when valuations are dependent, this condition is satisfied when valuations are suitably "positively dependent" (i.e., affiliated as in Milgrom (2004, Section 5.4.1), or alternatively, with the distribution of $\theta_{-i}|\theta_i$ monotonic in θ_i in the usual multivariate stochastic order), all players have correct beliefs (per Assumption 5) and use weakly increasing strategies (per Assumption 6), and *ex post* utility $\theta_i \overline{x}_i(a_i^*, a_{-i}) - \overline{t}_i(a_i^*, a_{-i})$ of player i weakly decreases with the actions of the other players, when player i takes the action $a_i^* = a_i(\theta_i; \theta_i'')$.

Lemma 2 (Sufficient conditions for Assumption 14)**.** *Suppose that for each $i \in \{1, 2, \ldots, N\}$, and any possible valuations θ_i and θ_i'', there is some selection*

$$a_i(\theta_i; \theta_i'') \in \arg\max_{a_i \in \mathcal{A}_i} \left(\theta_i E_{\Pi_i}(\overline{x}_i(a_i, a_{-i})|\theta_i'') - E_{\Pi_i}(\overline{t}_i(a_i, a_{-i})|\theta_i'')\right)$$

with $a_i(\theta_i; \theta_i) = a_i(\theta_i)$ from Assumption 6, such that for $a_i^* = a_i(\theta_i; \theta_i'')$, $\theta_i \overline{x}_i(a_i^*, a_{-i}) - \overline{t}_i(a_i^*, a_{-i})$ is a weakly decreasing function of a_{-i}. Suppose Assumptions 5 (Correct beliefs) and 6 (Weakly increasing strategy) are satisfied. Suppose either: (a) valuations are affiliated, or (b) the distribution of $\theta_{-i}|(\theta_i = \theta_i')$ is stochastically smaller than the distribution of $\theta_{-i}|(\theta_i = \theta_i'')$ in the usual multivariate stochastic order, when $\theta_i' \leq \theta_i''$. Then Assumption 14 is satisfied.

For example, in contest models from Example 1, the condition that *ex post* utility decreases with the actions of the other players is the intuitive condition that players are worse off when their opponents put forth more effort. Generically, in contest models, this would hold even without the restriction that player i takes the action a_i^* that maximizes "counterfactual" *ex interim* expected utility, as long as the "contest success function" for player i is a weakly decreasing function of the efforts of the other players. In other games, this restriction to using such an action a_i^* is important because if player i takes an extremely "irrational" action, then *ex post* utility of player i could be weakly *increasing* in the actions of the other players. For example, in a standard first price auction as a special case of Example 2, *ex post* utility is $(\theta_i - a_i)(1[a_i > \max_{j \neq i} a_j] + p_i(a) 1[a_i = \max_{j \neq i} a_j])$. If $a_i > \theta_i$, then *ex post* utility is weakly *increasing* in the actions of the other players, since player i is better off losing the auction since it overbid its valuation. But there is no incentive to bid more than its valuation, so in a first price auction $a_i^* \leq \theta_i$. For any such bid, player i is weakly worse off if the other players bid more.

Under the conditions of Lemma 2, if player i "counterfactually" has the beliefs of player i with valuation θ_i' with $\theta_i' \leq \theta_i''$ then player i believes the other players to take weakly lower actions compared to the case of having the beliefs of player i with valuation θ_i'', and therefore *ex interim* expected utility is weakly greater under "counterfactual" beliefs θ_i' compared to "counterfactual" beliefs θ_i'' since *ex post* utility is weakly greater when the actions of the other players are weakly lower. The conditions in Lemma 2 are sufficient but not necessary for Assumption 14, so a violation of these conditions does not imply that Assumption 14 fails. In particular, as noted above, Assumption 14 is satisfied with independent valuations, regardless of any other condition.

Equation 22 implies, under Assumptions 5 (Correct beliefs), 6 (Weakly increasing strategy), 13 (Counterfactual *ex interim* expected utility maximization problem has a solution), and 14 (Monotone effect of counterfactual beliefs on utility), for $\theta_i' < \theta_i < \theta_i''$, and letting $a_i(\theta_i; \theta_i'')$ be the selection per Assumption 14, for any $z_i \in \mathcal{A}_i$,

$$\begin{aligned}(26)\quad & \theta_i E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|\theta_i') - E_P(\overline{t}_i(a_i(\theta_i), A_{-i})|\theta_i') \geq \\ & \theta_i E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|\theta_i) - E_P(\overline{t}_i(a_i(\theta_i), A_{-i})|\theta_i) \geq \\ & \theta_i E_P(\overline{x}_i(a_i(\theta_i; \theta_i''), A_{-i})|\theta_i) - E_P(\overline{t}_i(a_i(\theta_i; \theta_i''), A_{-i})|\theta_i) \geq \\ & \theta_i E_P(\overline{x}_i(a_i(\theta_i; \theta_i''), A_{-i})|\theta_i'') - E_P(\overline{t}_i(a_i(\theta_i; \theta_i''), A_{-i})|\theta_i'') \geq \\ & \theta_i E_P(\overline{x}_i(z_i, A_{-i})|\theta_i'') - E_P(\overline{t}_i(z_i, A_{-i})|\theta_i'').\end{aligned}$$

Then, for any $z_i \in \mathcal{A}_i$, and letting $z_i' < a_i(\theta_i) < z_i''$ be any two actions that are actually used by player i, for at least some valuation of player i, i.e., $z_i', z_i'' \in \mathcal{A}_i^d$:

$$\begin{aligned}(27)\quad & \theta_i E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{t}_i(a_i(\theta_i), A_{-i})|A_i = z_i') \\ & = \theta_i E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|\theta_i' \in \Theta_i(z_i')) - E_P(\overline{t}_i(a_i(\theta_i), A_{-i})|\theta_i' \in \Theta_i(z_i')) \\ & \geq \theta_i E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|\theta_i) - E_{P_i}(\overline{t}_i(a_i(\theta_i), A_{-i})|\theta_i) \\ & \geq \theta_i E_P(\overline{x}_i(z_i, A_{-i})|\theta_i'' \in \Theta_i(z_i'')) - E_P(\overline{t}_i(z_i, A_{-i})|\theta_i'' \in \Theta_i(z_i'')) \\ & = \theta_i E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') - E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i'').\end{aligned}$$

And consequently,

(28)

$$\theta_i \geq \frac{(E_P(\overline{t}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i''))}{(E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i''))}$$
$$\forall z_i' < a_i(\theta_i) < z_i'', \{z_i', z_i''\} \in \mathcal{A}_i^d, z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') > 0\}$$
$$\theta_i \leq \frac{(E_P(\overline{t}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i''))}{(E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i''))}$$
$$\forall z_i' < a_i(\theta_i) < z_i'', \{z_i', z_i''\} \in \mathcal{A}_i^d, z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') < 0\}$$

Therefore, similarly to the identification strategy in Sections 3-4, structural identification of θ_i depends on game-structure identification of the expressions on the right hand side of Equation 28.

Definition 5 (Action with game-structure identification of differences)**.** A specification $(a_i, z_i, z_i', z_i'') \in (\mathcal{A}_i)^4$ of player i with $z_i', z_i'' \in \mathcal{A}_i^d$, is a specification with game-structure identification of differences if

$E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'')$ and $E_P(\overline{t}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i'')$

are point identified. The set of specifications with game-structure identification of differences is $\mathcal{R}_i$.

Establishing sufficient conditions for game-structure identification of differences is similar to establishing sufficient conditions for game-structure identification in Section 3. In particular, similarly to Section 3, even if the allocation rule and/or transfer rule are not known *ex ante* by the econometrician, $\overline{x}_i(a_i, a_{-i}) = E_P(X_i|A_i = a_i, A_{-i} = a_{-i})$ and $\overline{t}_i(a_i, a_{-i}) = E_P(T_i|A_i = a_i, A_{-i} = a_{-i})$ are point identified quantities under standard conditions on identification/estimation of conditional expectations. Then game-structure identification of differences can be established by taking the appropriate expectations (with respect to the distribution of the actions in the data) of the allocation rule and transfer rule displayed in Definition 5. Therefore, because of the similarity to results already reported in Section 3, in the interest of space, not all possible sufficient conditions are reported here. But of particular relevance to this extension of the identification strategy is the case of a discrete action space, so consider the case that $\mathcal{A}_i = \mathcal{A}_{i,\text{disc}}$ for all players i. Let $\mathcal{A}^d$ be the support of the observed actions $(A_1, A_2, \ldots, A_N)$, and let $\mathcal{A}_i^d$ be the support of the observed actions A_i. If $\mathcal{A}^d = \prod_i \mathcal{A}_i^d$, so that the support of $(A_1, A_2, \ldots, A_N)$ is the Cartesian product of the supports of the actions of each player, which is implied by the condition that the support of $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$ is the Cartesian product of the marginal supports of each θ_i,[51] then any specification of actions $(a_i, z_i, z_i', z_i'') \in (\mathcal{A}_i^d)^4$ of player i is a specification with game-structure identification of differences.[52]

Lemma 3 (Sufficient conditions for game-structure identification of differences with discrete actions)**.** *Suppose that Assumptions 1 (Dependent valuations) and 3 (Action space) are satisfied. Suppose the data is $P(A, X, T)$. If $\mathcal{A}_i = \mathcal{A}_{i,disc}$ for all players i, and $\mathcal{A}^d = \prod_i \mathcal{A}_i^d$, then any specification of actions $(a_i, z_i, z_i', z_i'') \in (\mathcal{A}_i^d)^4$ of player i is a specification with game-structure identification of differences per Definition 5. Consequently, $\mathcal{A}_i^d \times \mathcal{A}_i^d \times \mathcal{A}_i^d \times \mathcal{A}_i^d \subseteq \mathcal{R}_i$.*

Independent valuations *Under Assumption 2 (Independent valuations), A_{-i} is independent of A_i and therefore z_i' and z_i'' effectively play no role in Definition 5. So, under Assumption 2, a specification $(a_i, z_i) \in (\mathcal{A}_i)^2$ is a specification with game-structure identification of differences if it satisfies the condition in Definition 5, without the conditioning on z_i' and z_i''. Hence, under Assumption 2, the dimension of elements of $\mathcal{R}_i$ changes.* ★

An implication of Equation 28, restricted to specifications with game-structure identification of differences, is

$$
\begin{aligned}
(29)\qquad \theta_i &\geq \frac{(E_P(\overline{t}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i''))}{(E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i''))} \\
&\forall z_i' < a_i(\theta_i) < z_i'', z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') > 0\} \\
&(a_i(\theta_i), z_i, z_i', z_i'') \in \mathcal{R}_i \\
\theta_i &\leq \frac{(E_P(\overline{t}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i''))}{(E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i''))} \\
&\forall z_i' < a_i(\theta_i) < z_i'', z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i(\theta_i), A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') < 0\} \\
&(a_i(\theta_i), z_i, z_i', z_i'') \in \mathcal{R}_i
\end{aligned}
$$

[51]Suppose that $a_j^* \in \mathcal{A}_j^d$ for all players j. Then there must be θ_j^* in the support of θ_j such that $a_j^* = a_j(\theta_j^*)$. Hence, if the support of $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$ is the Cartesian product of the marginal supports of each θ_i, $(\theta_1^*, \theta_2^*, \ldots, \theta_N^*)$ is in the support of θ, so $a^* = (a_1^*, a_2^*, \ldots, a_N^*)$ is in the support of A.

[52]Since only differences in transfers are relevant for Definition 5, it is possible to accommodate two-part transfers similarly to Condition 5 of Lemma 1.

Let

(30)
$$\Phi_{Li}(a_i) = \max \begin{cases} \sup \begin{cases} \frac{\left(E_P(\bar{t}_i(a_i,A_{-i})|A_i=z_i') - E_P(\bar{t}_i(z_i,A_{-i})|A_i=z_i'')\right)}{\left(E_P(\overline{x}_i(a_i,A_{-i})|A_i=z_i') - E_P(\overline{x}_i(z_i,A_{-i})|A_i=z_i'')\right)} : \\ z_i' < a_i < z_i'', z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') > 0\}, \\ (a_i, z_i, z_i', z_i'') \in \mathcal{R}_i \end{cases} \\ \Theta_{Li} \end{cases}$$

and

(31)
$$\Phi_{Ui}(a_i) = \min \begin{cases} \inf \begin{cases} \frac{\left(E_P(\bar{t}_i(a_i,A_{-i})|A_i=z_i') - E_P(\bar{t}_i(z_i,A_{-i})|A_i=z_i'')\right)}{\left(E_P(\overline{x}_i(a_i,A_{-i})|A_i=z_i') - E_P(\overline{x}_i(z_i,A_{-i})|A_i=z_i'')\right)} : \\ z_i' < a_i < z_i'', z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') < 0\}, \\ (a_i, z_i, z_i', z_i'') \in \mathcal{R}_i \end{cases} \\ \Theta_{Ui} \end{cases}$$

where Θ_{Li} and Θ_{Ui} are *ex ante* known bounds on valuations from Assumption 8. Consequently, the valuation corresponding to a_i must be between $\Phi_{Li}(a_i)$ and $\Phi_{Ui}(a_i)$.

Independent valuations *Under Assumption 2 (Independent valuations), but even without Assumptions 6 (Weakly increasing strategy), 13 (Counterfactual* ex interim *expected utility maximization problem has a solution), and 14 (Monotone effect of counterfactual beliefs on utility), based on similar steps, the θ_i consistent with a given observed action a_i is in the set $\Xi_i(a_i) = [\Xi_{Li}(a_i), \Xi_{Ui}(a_i)]$ with*

$$\Xi_{Li}(a_i) = \max \begin{cases} \sup \begin{cases} \frac{\left(E_P(\bar{t}_i(a_i,A_{-i})) - E_P(\bar{t}_i(z_i,A_{-i}))\right)}{\left(E_P(\overline{x}_i(a_i,A_{-i})) - E_P(\overline{x}_i(z_i,A_{-i}))\right)} : \\ z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i, A_{-i})) - E_P(\overline{x}_i(z_i, A_{-i})) > 0\}, (a_i, z_i) \in \mathcal{R}_i \end{cases} \\ \Theta_{Li} \end{cases} \tag{32}$$

and

$$\Xi_{Ui}(a_i) = \min \begin{cases} \inf \begin{cases} \frac{\left(E_P(\bar{t}_i(a_i,A_{-i})) - E_P(\bar{t}_i(z_i,A_{-i}))\right)}{\left(E_P(\overline{x}_i(a_i,A_{-i})) - E_P(\overline{x}_i(z_i,A_{-i}))\right)} : \\ z_i \in \{\mathcal{A}_i : E_P(\overline{x}_i(a_i, A_{-i})) - E_P(\overline{x}_i(z_i, A_{-i})) < 0\}, (a_i, z_i) \in \mathcal{R}_i \end{cases} \\ \Theta_{Ui} \end{cases} \tag{33}$$

Further, under Assumption 2 (Independent valuations), "one-step" game-structure identification of differences can be established similarly to Section 3: $E_P(\bar{t}_i(a_i, A_{-i})) = E_P(T_i|A_i = a_i)$, $E_P(\overline{x}_i(a_i, A_{-i})) = E_P(X_i|A_i = a_i)$, $E_P(\bar{t}_i(z_i, A_{-i})) = E_P(T_i|A_i = z_i)$, and $E_P(\overline{x}_i(z_i, A_{-i})) = E_P(X_i|A_i = z_i)$. ★

Then, by Assumption 6 (Weakly increasing strategy) and arguments similar to those in Section 3, any valuation consistent with a_i is between $\sup_{a_i' \leq a_i, a_i' \in \mathcal{A}_i^d} \Phi_{Li}(a_i')$ and $\inf_{a_i' \geq a_i, a_i' \in \mathcal{A}_i^d} \Phi_{Ui}(a_i')$. Let

(34)
$$\Upsilon_{Li}(a_i) = \max\{\sup_{a_i' \leq a_i, a_i' \in \mathcal{A}_i^d} \Phi_{Li}(a_i'), \sup_{a_i' \in \rho_{Li}(a_i)} \Psi_i(a_i')\} \text{ and } \Upsilon_{Ui}(a_i) = \min\{\inf_{a_i' \geq a_i, a_i' \in \mathcal{A}_i^d} \Phi_{Ui}(a_i'), \inf_{a_i' \in \rho_{Ui}(a_i)} \Psi_i(a_i')\}.$$

The identification strategy from Theorem 1 still applies, so the valuation θ_i corresponding to action a_i is bounded between $\Upsilon_{Li}(a_i)$ and $\Upsilon_{Ui}(a_i)$. However, under various conditions, for various actions a_i, the identification due to Section 3 or the identification due to this Appendix D can be the *only* relevant source of identification. Recall that if the action space is entirely discrete, or *ex interim* expected allocation and/or *ex interim* expected transfer is nowhere differentiable, then the identification result from Section 3 are the trivial bounds that valuations lie between Θ_{Li} and Θ_{Ui}. Under those conditions, obviously the identification due to this Appendix D is the only relevant source of identification for any action a_i. Indeed, those conditions were the motivation for developing the extension of the identification strategy in this section. Conversely, if a_i is part of the continuous part of the action space, and satisfies the other conditions for point identification of the corresponding valuation θ_i based on the identification result in Section 3, then obviously the identification due to Section 3 is the only relevant source of identification for that action a_i. Hence, the additional Assumptions 13-14 are not necessary for the identification result relative to that action a_i. Moreover, if the conditions of Theorem 2 hold, then there is point identification of the distribution of valuations, in which case obviously the addition of Assumptions 13-14 has no effect on the identification result.

Independent valuations *Also let*

(35)

$$\Gamma_{Li}(a_i) = \max\{\sup_{a'_i \leq a_i, a'_i \in \mathcal{A}^d_i} \Xi_{Li}(a'_i), \sup_{a'_i \in \rho_{Li}(a_i)} \Lambda_i(a'_i)\} \text{ and } \Gamma_{Ui}(a_i) = \min\{\inf_{a'_i \geq a_i, a'_i \in \mathcal{A}^d_i} \Xi_{Ui}(a'_i), \inf_{a'_i \in \rho_{Ui}(a_i)} \Lambda_i(a'_i)\}.$$

Theorem 4. *Under Assumptions 1 (Dependent valuations), 3 (Action space), 4 (Optimal strategy), 5 (Correct beliefs), 6 (Weakly increasing strategy), 8 (Known bounds on valuations), 13 (Counterfactual* ex interim *expected utility maximization problem has a solution), and 14 (Monotone effect of counterfactual beliefs on utility), the distribution of valuations θ is partially identified, and the identification is constructive, because the distribution of θ is stochastically larger than the distribution of $(\Upsilon_{L1}(A_1), \Upsilon_{L2}(A_2), \ldots, \Upsilon_{LN}(A_N))$ and is stochastically smaller than the distribution of $(\Upsilon_{U1}(A_1), \Upsilon_{U2}(A_2), \ldots, \Upsilon_{UN}(A_N))$, in the sense of the usual multivariate stochastic order, where $(A_1, A_2, \ldots, A_N)$ is distributed according to the data $P(A, X, T)$ and $\Upsilon_{Li}(\cdot)$ and $\Upsilon_{Ui}(\cdot)$ are the identifiable functions given in Equation 34 (see Lemmas 1 and 3).*

Independent valuations *With independent valuations: replace Assumption 1 (Dependent valuations) with Assumption 2 (Independent valuations), drop Assumption 6 (Weakly increasing strategy), 13 (Counterfactual* ex interim *expected utility maximization problem has a solution), and 14 (Monotone effect of counterfactual beliefs on utility), add Assumption 7 (Non-decreasing expected allocation rule), replace the definition of $\tilde{\mathcal{A}}^d_i$ from Equation 17 with $\tilde{\mathcal{A}}^d_i = \{a'_i \in \mathcal{A}^d_i \cap \mathcal{A}_{i,cont} : \Lambda^x_i(a'_i)$ exists and $\Lambda^t_i(a'_i)$ exists and $\Lambda^x_i(a'_i) > 0$ and a'_i is a point of game-structure identification per Definition 4$\}$, and replace the Υ functions with the Γ functions defined in Equation 35.* ★

Remark 2 (Connection to identification strategy in Section 3)**.** The identification strategy in this section is a "discrete analogue" of the identification strategy in Section 3. One main difference is that the identification strategy in this section takes a different approach to *bounding* the beliefs of the players. This is necessary because the approach to dealing with the beliefs of the players used in Section 3 cannot apply to games with discrete action spaces. To see the relationship between the identification strategies, based on a heuristic/intuitive argument in which the action space does include a "continuous part," suppose $a_i \in \text{int}(\mathcal{A}_{i,\text{cont}})$. Then, consider the limit of $z_i \to a_i, z_i' \uparrow a_i, z_i'' \downarrow a_i$ in the right hand sides of Equations 30-31. Under appropriate conditions and assumptions, the resulting limit is the same ratio of derivatives that formed the identification strategy in Section 3. In order for such a limit to make sense, a_i must be in the continuous part of the action space. And since $z_i' < a_i < z_i''$ in the right hand sides of Equations 30-31, a_i must be in the *interior* of the action space. This limit can also approximate a (heuristic) limit when the discrete part of the action space with increasingly many actions becomes a continuous/interval action space, with the substantial caveat that the game itself changes when the action space changes, so such a limit cannot be taken literally without a careful analysis of how the game changes.

Under the appropriate conditions and assumptions, $\frac{\left(E_P(\bar{t}_i(a_i,A_{-i})|A_i=z_i')-E_P(\bar{t}_i(z_i,A_{-i})|A_i=z_i'')\right)}{\left(E_P(\overline{x}_i(a_i,A_{-i})|A_i=z_i')-E_P(\overline{x}_i(z_i,A_{-i})|A_i=z_i'')\right)} \overset{z_i'\uparrow a_i \text{ and } z_i''\downarrow a_i}{\longrightarrow}$ $\frac{\left(E_P(\bar{t}_i(a_i,A_{-i})|A_i=a_i)-E_P(\bar{t}_i(z_i,A_{-i})|A_i=a_i)\right)}{(E_P(\overline{x}_i(a_i,A_{-i})|A_i=a_i)-E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i))} = \frac{\frac{(E_P(\bar{t}_i(a_i,A_{-i})|A_i=a_i)-E_P(\bar{t}_i(z_i,A_{-i})|A_i=a_i))}{a_i-z_i}}{\frac{(E_P(\overline{x}_i(a_i,A_{-i})|A_i=a_i)-E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i))}{a_i-z_i}} \overset{z_i\to a_i}{\longrightarrow} \frac{\left.\frac{\partial E_P(\bar{t}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i=a_i}}{\left.\frac{\partial E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i=a_i}}$.

The first limit requires continuity of the conditional expectations as a function of the conditioning variable, so that $E_P(\bar{t}_i(a_i,A_{-i})|A_i = z_i') \to E_P(\bar{t}_i(a_i,A_{-i})|A_i = a_i)$ and $E_P(\overline{x}_i(a_i,A_{-i})|A_i = z_i') \to E_P(\overline{x}_i(a_i,A_{-i})|A_i = a_i)$ as $z_i' \uparrow a_i$ and $E_P(\bar{t}_i(z_i,A_{-i})|A_i = z_i'') \to E_P(\bar{t}_i(z_i,A_{-i})|A_i = a_i)$ and $E_P(\overline{x}_i(z_i,A_{-i})|A_i = z_i'') \to E_P(\overline{x}_i(z_i,A_{-i})|A_i = a_i)$ as $z_i'' \downarrow a_i$, where the third and fourth limits must hold uniformly over z_i since z_i is part of the limiting sequence.[53] The second limit is an application of the definition of the derivative, and requires that the derivatives exist and that $\left.\frac{\partial E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i=a_i} \neq 0$. These conditions are closely related to (and slightly stronger than) the conditions for point identification discussed after Theorem 1 and used in Theorem 2. In that case, the valuation θ_i corresponding to action a_i is bounded above and below by, and thus must equal, $\frac{\left.\frac{\partial E_P(\bar{t}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i=a_i}}{\left.\frac{\partial E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i=a_i}} = \Psi_i(a_i)$.[54] Hence, based on this heuristic/intuitive argument, this is closely related to the identification result established in Section 3, under similar assumptions, showing

[53]Continuity of the conditional expectations is related to the condition of no point masses used in Section 3. Suppose $a_i(\theta_i) = a_i^*$ has the unique solution θ_i^*, so θ_i^* is the unique valuation to use action a_i^*. Then there will be no point mass at a_i^* in the distribution of A_i. Suppose further that $a_i(\cdot)$ is *strictly* increasing in a neighborhood of θ_i^*, and that $a_i(\cdot)$ is continuous in a neighborhood of θ_i^*. The first condition is slightly stronger than the condition that θ_i^* is the unique valuation to use action a_i^*, since it could otherwise be that, for example, $a_i(\cdot)$ is strictly increasing "below" θ_i^*, has a jump discontinuity at θ_i^*, and is flat "above" θ_i^*. Since $a_i(\cdot)$ is weakly increasing per Assumption 6, $a_i(\cdot)$ is continuous except for a countable set. Then, for example, $E_P(\bar{t}_i(a_i,A_{-i})|A_i = z_i') = E_P(\bar{t}_i(a_i,A_{-i})|\theta_i = a_i^{-1}(z_i'))$. Supposing that $E_P(\bar{t}_i(a_i,A_{-i})|\theta_i) = E_{\Pi_i}(\bar{t}_i(a_i,a_{-i})|\theta_i)$ is itself continuous as a function of θ_i, which could be established using economic theory similar to related discussion in Section 3 , it would follow that $E_P(\bar{t}_i(a_i,A_{-i})|A_i = z_i') \to E_P(\bar{t}_i(a_i,A_{-i})|A_i = a_i)$ as $z_i' \to a_i$ and similarly for the other limits of the other conditional expectations. Otherwise, if there were multiple valuations to use action a_i, resulting in a point mass at a_i, a "small change" in conditioning on $A_i = a_i$ versus $A_i = z_i'$ could result in a "large change" in the actual expected value, since it would correspond to a "large change" in the set of θ_i being equivalently conditioned on.

[54]This heuristic analysis also implicitly assumes game-structure identification on the right hand side of Equations 30-31. Further, under the condition that $\left.\frac{\partial E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i=a_i} \neq 0$, assume that $\left.\frac{\partial E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i}$ is continuous in z_i (i.e., continuously differentiable). Consider the case that $\left.\frac{\partial E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i} > 0$ on an interval neighborhood of a_i. The case that $\left.\frac{\partial E_P(\overline{x}_i(z_i,A_{-i})|A_i=a_i)}{\partial z_i}\right|_{z_i} < 0$ on an interval neighborhood of a_i would be similar, except as discussed above, seems inconsistent with Assumption 6. Then $E_P(\overline{x}_i(z_i,A_{-i})|A_i = a_i)$ would be strictly increasing at $z_i = a_i$, and hence (when $z_i' \approx a_i \approx z_i''$), $z_i < a_i$ would generally satisfy the condition that $E_P(\overline{x}_i(a_i,A_{-i})|A_i =$

the sense in which the identification strategy in this section is a sort of "discrete analogue" of the identification strategy in Section 3. More broadly, viewing this limit as a (heuristic) limit when the discrete part of the action space with increasingly many actions becomes a continuous/interval action space, this suggests that relatively finer discrete action spaces (e.g., auctions that allow bids that are any multiple of one cent compared to any multiple of five dollars) can be expected to result in relatively tighter identification of the distribution of valuations, with the limit in Section 3.

Appendix E. Proofs

In order to economize on space, references to equations and other quantities already defined in the body of the paper are used in the proofs.

Proof of Lemma 1. Condition 1: The definitions of $\Psi_i^x(\cdot)$ and $\Psi_i^t(\cdot)$ are given in Equation 8. By definition, $\overline{x}_i(a) = E(\widetilde{x}_i(a)) = E(\widetilde{x}_i(a)|A_i = a_i, A_{-i} = a_{-i}) = E(X_i|A_i = a_i, A_{-i} = a_{-i})$ and $\overline{t}_i(a) = E(\widetilde{t}_i(a)) = E(\widetilde{t}_i(a)|A_i = a_i, A_{-i} = a_{-i}) = E(T_i|A_i = a_i, A_{-i} = a_{-i})$. Therefore, by substitution, the expressions in Equation 13 are valid. Let $a_i \in \mathcal{A}_i^d$ be given, and let $\mathcal{S}$ be given with the properties in the statement of Condition 1. Let $a_i' \in \mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{S}$. By assumption, $E_P(X_i|A_i = a_i', A_{-i} = a_{-i})$ and $E_P(T_i|A_i = a_i', A_{-i} = a_{-i})$ are point identified for all a_{-i} in a probability 1 subset of the support of $A_{-i}|(A_i = a_i)$. Therefore, given that the distribution of $A_{-i}|(A_i = a_i)$ is point identified by assumption, $E_P(E_P(X_i|A_i = a_i', A_{-i})|A_i = a_i)$ and $E_P(E_P(T_i|A_i = a_i', A_{-i})|A_i = a_i)$ are point identified. Consequently, the existence and values of $\Psi_i^x(a_i)$ and $\Psi_i^t(a_i)$ are point identified by the existence and values of the limits corresponding to expressions in Equation 13.

Condition 2: The proof involves establishing that Condition 1 holds. The first step is to show $A_{-i}|(A_i = a_i)$ is point identified. Consider $a_i^{-1}(\tilde{\tilde{\mathcal{A}}}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{I}) = \{\theta_i : a_i(\theta_i) \in \tilde{\tilde{\mathcal{A}}}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{I}\}$. Since $\tilde{\tilde{\mathcal{A}}}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{I}$ is a non-degenerate interval, $a_i^{-1}(\tilde{\tilde{\mathcal{A}}}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{I})$ is a non-degenerate interval since $a_i(\cdot)$ is weakly increasing per Assumption 6. Consider $\theta_i' < \theta_i''$ in $a_i^{-1}(\tilde{\tilde{\mathcal{A}}}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{I})$. By Assumption 6, $a_i(\theta_i') \leq a_i(\theta_i'')$. Moreover, by the same arguments as used in the proof of Theorem 1, if $a_i(\theta_i') = a_i(\theta_i'')$ then there is a point mass in the distribution of A_i located at $a_i(\theta_i') = a_i(\theta_i'')$, a contradiction since $a_i(\theta_i') = a_i(\theta_i'') \in \tilde{\tilde{\mathcal{A}}}_i^d$ by construction. Therefore, $a_i(\cdot)$ is strictly increasing on $a_i^{-1}(\tilde{\tilde{\mathcal{A}}}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{I})$. Let θ_i^* be the unique valuation to use action a_i, since $a_i \in \tilde{\tilde{\mathcal{A}}}_i^d$. Then since $\theta_i^* \in a_i^{-1}(\tilde{\tilde{\mathcal{A}}}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{I})$ by construction, $a_i(\cdot)$ is strictly increasing on an interval containing θ_i^* and therefore has an inverse on an interval containing a_i. Note that $a_i(a_i^{-1}(\tilde{\tilde{\mathcal{A}}}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{I})) = \tilde{\tilde{\mathcal{A}}}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{I}$, since by construction $\tilde{\tilde{\mathcal{A}}}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{I}$ is in the image of $a_i(\cdot)$. Because $a_i(\cdot)$ is strictly increasing on the interval $a_i^{-1}(\tilde{\tilde{\mathcal{A}}}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{I})$ with associated image the interval $\tilde{\tilde{\mathcal{A}}}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{I}$, $a_i(\cdot)$ is continuous on $a_i^{-1}(\tilde{\tilde{\mathcal{A}}}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{I})$ since monotone functions can only have jump discontinuities (or see for example Ghorpade and Limaye (2006, Section 3.2)). Because $a_i(\cdot)$ is continuous on the interval $a_i^{-1}(\tilde{\tilde{\mathcal{A}}}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{I})$, the inverse on the interval $\tilde{\tilde{\mathcal{A}}}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{I}$ is continuous. Therefore, for $a_i' \in \tilde{\tilde{\mathcal{A}}}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{I}$, $A_{-i}|(A_i = a_i') = A_{-i}|(\theta_i = a_i^{-1}(a_i'))$. Under Assumption 6, $A_{-i}|(\theta_i = a_i^{-1}(a_i')) = a_{-i}(\theta_{-i})|(\theta_i = a_i^{-1}(a_i'))$. Because $a_j(\cdot)$ are weakly increasing functions per Assumption 6, the set of θ_j such that $a_j(\theta_j) \leq t_j$ is an interval. Therefore, the boundary of the set of

$z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') > 0$ in the right hand side of Equation 30 and $z_i > a_i$ would generally satisfy the condition that $E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i') - E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'') < 0$ in the right hand side of Equation 31.

θ_{-i} such that $a_{-i}(\theta_{-i}) \leq t_{-i}$ has probability zero under the continuous distribution of $\theta_{-i}|(\theta_i = \theta_i^*)$. Consequently, $P(a_{-i}(\theta_{-i}) \leq t_{-i}|\theta_i = \theta_i')$ converges to $P(a_{-i}(\theta_{-i}) \leq t_{-i}|\theta_i = \theta_i^*)$ if $\theta_i' \to \theta_i^*$, since $\theta_{-i}|(\theta_i = \theta_i')$ converges weakly to $\theta_{-i}|(\theta_i = \theta_i^*)$ by assumption. That holds because the condition that the density of $\theta_{-i}|(\theta_i = \theta_i')$ converges everywhere to the density of $\theta_{-i}|(\theta_i = \theta_i^*)$ as $\theta_i' \to \theta_i^*$ implies that $\theta_{-i}|\theta_i'$ converges in total variation (and hence weakly) to $\theta_{-i}|\theta_i^*$ by Scheffé (1947)'s lemma. Therefore, as a weaker conclusion, $A_{-i}|(A_i = a_i') \to^w A_{-i}|(A_i = a_i)$ as $a_i' \to a_i$ with $a_i' \in \tilde{\tilde{\mathcal{A}}}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{I}$, so $A_{-i}|(A_i = a_i)$ is point identified. That follows since $P(A_{-i} \leq t_{-i}|A_i = a_i') = E_P(1[A_{-i} \leq t_{-i}]|A_i = a_i')$. Therefore, equivalently, $E_P(1[A_{-i} \leq t_{-i}]|A_i = a_i') \to E_P(1[A_{-i} \leq t_{-i}]|A_i = a_i)$ as $a_i' \to a_i$ with $a_i' \in \tilde{\tilde{\mathcal{A}}}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{I}$, for t_{-i} a continuity point of the distribution $A_{-i}|(A_i = a_i)$. Hence, since conditional expectations are point identified at points of continuity of the conditioning variable, $E_P(1[A_{-i} \leq t_{-i}]|A_i = a_i)$ is point identified and therefore $P(A_{-i} \leq t_{-i}|A_i = a_i)$ is point identified, when t_{-i} is a continuity point, and therefore the distribution $A_{-i}|(A_i = a_i)$ is point identified.

The second step is to show the set $\mathcal{S}$ exists. By the continuity assumption on $\overline{x}_i(a_i', a_{-i})$ and $\overline{t}_i(a_i', a_{-i})$, $E_P(X_i|A_i = a_i', A_{-i} = a_{-i})$ and $E_P(T_i|A_i = a_i', A_{-i} = a_{-i})$ are point identified for all $a_i' \in \mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{N}$ and $a_{-i} \in \tilde{\mathcal{A}}_{-i}^d(a_i')$. Suppose that $a_i \in (\text{int}(\mathcal{A}_{i,\text{cont}}) \cap \text{bd}(\mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}}))^C \cap (\mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}}) = ((\text{int}(\mathcal{A}_{i,\text{cont}}))^C \cup (\text{bd}(\mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}}))^C) \cap (\mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}})$. By assumption, $\tilde{\tilde{\mathcal{A}}}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{I}$ is a non-degenerate interval that contains a_i, so $\mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{N}$ is a non-degenerate interval that contains a_i, since $\mathcal{N}$ is a neighborhood of $a_i \in \mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}}$. Further, if $a_i \in \text{int}(\mathcal{A}_{i,\text{cont}})$, then $a_i \in (\text{bd}(\mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}}))^C \cap (\mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}}) = \text{int}(\mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}})$. Therefore, there is a neighborhood of a_i that is contained in $\mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}}$. And therefore $\mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{N}$ is a neighborhood of a_i and $a_i \in \text{int}(\mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{N})$. Therefore, $\mathcal{S} = \mathcal{N}$ satisfies the statement of Condition 1.

Condition 3: The definitions of $\Lambda_i^x(\cdot)$ and $\Lambda_i^t(\cdot)$ are given in Equation 14. By the arguments of Footnote 27, the expressions in Equation 16 are valid. By the assumption on $\mathcal{A}_i^d \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{S}$ in the statement of the condition, the existence and values of $\Lambda_i^x(a_i)$ and $\Lambda_i^t(a_i)$ are point identified by the existence and values of the limits corresponding to expressions in Equation 16.

Condition 4: The definitions of $\Psi_i^x(\cdot)$ and $\Psi_i^t(\cdot)$ are given in Equation 8. Let $a_i \in \mathcal{A}_i^d$ be given, and let $\mathcal{S}$ be given with the properties in the statement of the condition. Let $a_i' \in \mathcal{A}_i \cap \mathcal{A}_{i,\text{cont}} \cap \mathcal{S}$. By assumption, $\overline{x}_i(a_i', a_{-i})$ and $\overline{t}_i(a_i', a_{-i})$ are *ex ante* known by the econometrician for all a_{-i} in a probability 1 subset of the support of $A_{-i}|(A_i = a_i)$. Therefore, given that the distribution of $A_{-i}|(A_i = a_i)$ is point identified by assumption, $E_P(\overline{x}_i(a_i', a_{-i})|A_i = a_i)$ and $E_P(\overline{t}_i(a_i', a_{-i})|A_i = a_i)$ are point identified. Consequently, the existence and values of $\Psi_i^x(a_i)$ and $\Psi_i^t(a_i)$ are point identified by the existence and values of the limits corresponding to expressions in Equation 8.

Condition 5: Under these conditions, evaluated at any z satisfying the conditions on a_i in the statement of the Conditions, $\Psi_i^t(z) \equiv \frac{\partial E_P(\overline{t}_i(a_i, A_{-i})|A_i = z)}{\partial a_i}\Big|_{a_i = z} = \frac{\partial E_P(\overline{t}_{i1}(a_i, A_{-i})|A_i = z)}{\partial a_i}\Big|_{a_i = z}$. Therefore, if any of Conditions 1-4 hold with t_{i1} in place of t_i and T_{i1} in place of T_i, the result holds. □

Proof of Theorem 1. By Assumptions 3 and 4, Equations 2-4 are necessary conditions. See Footnote 18 for the arguments. By Assumptions 1 and 6, conditioning on θ_i is equivalent to conditioning on $A_i = a_i(\theta_i)$, if θ_i is the unique valuation to use action $a_i(\theta_i)$.[55] By the arguments of Footnote 24, the

[55] As a technical note, it is worth observing that conditioning on $a_i^* \in \mathcal{A}_i^d$ that is used by a unique valuation θ_i^* is indeed equivalent to conditioning on the unique associated θ_i^* under the regularity conditions implied by this setup.

set of θ_i such that $a_i(\theta_i) = a_i^*$ is an interval. Consequently, if two distinct valuations θ_i and θ_i' use action a_i^* then the non-degenerate interval containing θ_i and θ_i' uses action a_i^*. Under Assumption 1, the probability of that interval of valuations is strictly positive, implying a point mass at $A_i = a_i^*$ in the data. By the contrapositive, if there is not a point mass at $A_i = a_i^*$ in the data, then a_i^* is used by a unique valuation θ_i. Therefore, under Assumptions 1, 5, and 6, Equations 5-7 are valid. Consequently, Equations 10-12 are valid.

Let $a_i \in \mathcal{A}_i^d$ be given. Consider any $a_i' \in \rho_{Li}(a_i)$, defined in Equation 18. By construction of the properties of a_i', the above paragraph applies to a_i', so player i that uses action a_i' has valuation θ_i' with identification according to Equations 10-12. In particular, if a_i' is not on the upper bound of $\mathcal{A}_{i,\text{cont}}$, then the valuation θ_i' corresponding to the use of action a_i' is point identified according to Equation 10. Alternatively, if a_i' is on the upper bound of $\mathcal{A}_{i,\text{cont}}$, then the valuation θ_i' corresponding to the use of action a_i' can be provided a lower bound according to Equation 12. Therefore, overall, the valuation θ_i' corresponding to a_i' satisfies $\theta_i' \geq \Psi_i(a_i')$.

Consider any $\tilde{\theta}_i < \Psi_i(a_i')$ with $a_i' \in \rho_{Li}(a_i)$. If θ_i' is any valuation consistent with using action a_i', then $\theta_i' \geq \Psi_i(a_i')$. Moreover, since $a_i' \in \mathcal{A}_i^d$ by construction, there is indeed some valuation θ_i' that uses action a_i'. By Assumption 6, the action used by valuation $\tilde{\theta}_i$ is weakly less than the action used by valuation $\theta_i' \geq \Psi_i(a_i') > \tilde{\theta}_i$, so the action used by valuation $\tilde{\theta}_i$ is weakly less than a_i'. Moreover, since $\tilde{\theta}_i \ngeq \Psi_i(a_i')$ by construction, valuation $\tilde{\theta}_i$ cannot use action a_i'. Consequently, player i with valuation $\tilde{\theta}_i$ must use an action strictly less than a_i'. By the contrapositive, any action weakly greater than a_i' must correspond to a valuation weakly greater than $\Psi_i(a_i')$. Consequently, because $a_i' \leq a_i$, the valuation θ_i corresponding to the use of action a_i must be weakly greater than $\Psi_i(a_i')$.

Since the above holds for any $a_i' \in \rho_{Li}(a_i)$, the valuation θ_i corresponding to the use of action a_i must be weakly greater than $\sup_{a_i' \in \rho_{Li}(a_i)} \Psi_i(a_i')$. By similar arguments, the valuation θ_i corresponding to the use of action a_i must be weakly less than $\inf_{a_i' \in \rho_{Ui}(a_i)} \Psi_i(a_i')$. Because the valuation must

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Thus, conditioning on logically equivalent sets of probability zero are the same conditional quantity. The following works out the details. If $a_i^* \in \mathcal{A}_i^d$ and a_i^* is an isolated point of $\mathcal{A}_i^d$, then a_i^* is a mass point. That follows since the probability of any neighborhood of a_i^* is strictly positive by definition of support. Since a_i^* is an isolated point, there is a neighborhood of a_i^* that has intersection $\{a_i^*\}$ with $\mathcal{A}_i^d$. Therefore, a_i^* itself must have positive probability, i.e., is a mass point. Hence, any $a_i^* \in \mathcal{A}_i^d$ that is not a mass point is not an isolated point of $\mathcal{A}_i^d$. Consider any such a_i^*. Let θ_i^* be the unique valuation such that $a_i^* = a_i(\theta_i^*)$. Since $a_i(\cdot)$ is a weakly increasing function by Assumption 6, $\lim_{\theta_i \uparrow \theta_i^*} a_i(\theta_i)$ and $\lim_{\theta_i \downarrow \theta_i^*} a_i(\theta_i)$ both exist, if θ_i^* is in the interior of the support of θ_i. Otherwise only one such limit can be defined. By monotonicity, it must be that $\lim_{\theta_i \uparrow \theta_i^*} a_i(\theta_i) \leq a_i(\theta_i^*) = a_i^* \leq \lim_{\theta_i \downarrow \theta_i^*} a_i(\theta_i)$, for the limits that are defined. If $\lim_{\theta_i \uparrow \theta_i^*} a_i(\theta_i) < a_i(\theta_i^*) = a_i^* < \lim_{\theta_i \downarrow \theta_i^*} a_i(\theta_i)$, then a_i^* would be an isolated point of $\mathcal{A}_i^d$, a contradiction. So either $\lim_{\theta_i \uparrow \theta_i^*} a_i(\theta_i) = a_i(\theta_i^*) = a_i^*$ or $a_i(\theta_i^*) = a_i^* = \lim_{\theta_i \downarrow \theta_i^*} a_i(\theta_i)$. Therefore, $a_i(\cdot)$ is either left- or right- continuous at θ_i^*. Consider the case that $\lim_{\theta_i \uparrow \theta_i^*} a_i(\theta_i) = a_i(\theta_i^*) = a_i^*$. The other case is similar. Define $\hat{\theta}_i(a_i) = \inf_{\theta_i} \{\theta_i : a_i(\theta_i) \geq a_i\}$. Because $a_i(\cdot)$ is a weakly increasing function, for any $a_i' \leq a_i^*$, the set $\{\theta_i : a_i' \leq a_i(\theta_i) \leq a_i^*\}$ is the same as the set $\{\theta_i : \hat{\theta}_i(a_i') \leq \theta_i \leq \theta_i^*\}$, up to possibly the lower inequality being strict if $a_i(\hat{\theta}_i(a_i')) < a_i'$. Since θ_i has a continuous distribution per Assumption 1, that does not affect the probability of the set. Further, the former set is the same event, by construction, as $\{A_i : a_i' \leq A_i \leq a_i^*\}$. By construction, $\hat{\theta}_i(a_i^*) = \theta_i^*$, since $a_i(\theta_i^*) = a_i^*$ and any $\theta_i < \theta_i^*$ has $a_i(\theta_i) < a_i(\theta_i^*)$ since θ_i^* is the unique valuation to use a_i^*. And, $\hat{\theta}_i(a_i') < \theta_i^*$ for $a_i' < a_i^*$, since there must be $\theta_i'' < \theta_i^*$ such that $a_i' < a_i(\theta_i'')$ by left-continuity. Hence, $\hat{\theta}_i(a_i') < \theta_i^*$. Moreover, let $\theta_i'' < \theta_i^*$ be given with associated actions $a_i(\theta_i'') < a_i^*$. By left-continuity at θ_i^*, and weakly increasing, there is some θ_i' such that $\theta_i'' < \theta_i' < \theta_i^*$ and $a_i(\theta_i'') < a_i(\theta_i') < a_i^*$, and therefore $\hat{\theta}(a_i(\theta_i')) \geq \theta_i''$ since $a_i(\theta_i'') < a_i(\theta_i')$. Hence, $\hat{\theta}_i(a_i') \uparrow \theta_i^*$ as $a_i' \uparrow a_i^*$. Therefore conditioning on $A_i = a_i^*$ for $a_i^* \in \mathcal{A}_i^d$ is equivalent to conditioning on the unique associated θ_i^* per the definition of conditional probability, since it satisfies the standard definition of either corresponding to a mass point or the limit of decreasing neighborhoods of the conditioning event.

be between Θ_{Li} and Θ_{Ui}, by Assumption 8, the valuation corresponding to action a_i must be between $\kappa_{Li}(a_i)$ and $\kappa_{Ui}(a_i)$. Therefore, considering the joint distribution of $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$ and corresponding observed actions $A = (A_1, A_2, \ldots, A_N)$, it holds for all realizations that, for each $i \in \{1, 2, \ldots, N\}$, $\kappa_{Li}(A_i) \leq \theta_i \leq \kappa_{Ui}(A_i)$. Consequently, the partial identification result in the usual multivariate stochastic order follows from Shaked and Shanthikumar (2007, Theorem 6.B.1).

Independent valuations *Under Assumption 2, the following adjustments are made to the proof. Equations 2-4 need not condition on θ_i since beliefs are independent of valuation. Thus, Equations 5-7 are valid without conditioning on A_i, so Assumption 6 need not be used, and the restriction to $\tilde{\tilde{\mathcal{A}}}_i^d$ is not necessary.*

Assumption 7 implies that ex interim *expected utility satisfies the single crossing property per the following. If $V_i(a_i', \theta_i'') > V_i(a_i'', \theta_i'')$ with $a_i' > a_i''$ and $\theta_i' > \theta_i''$, then $\theta_i'' E_{\Pi_i}(\overline{x}_i(a_i', a_{-i})) - E_{\Pi_i}(\overline{t}_i(a_i', a_{-i})) > \theta_i'' E_{\Pi_i}(\overline{x}_i(a_i'', a_{-i})) - E_{\Pi_i}(\overline{t}_i(a_i'', a_{-i}))$. Consequently, $\theta_i''(E_{\Pi_i}(\overline{x}_i(a_i', a_{-i})) - E_{\Pi_i}(\overline{x}_i(a_i'', a_{-i}))) > E_{\Pi_i}(\overline{t}_i(a_i', a_{-i})) - E_{\Pi_i}(\overline{t}_i(a_i'', a_{-i}))$. By Assumption 7, $E_{\Pi_i}(\overline{x}_i(a_i', a_{-i})) - E_{\Pi_i}(\overline{x}_i(a_i'', a_{-i})) \geq 0$. Consequently, since $\theta_i' > \theta_i''$, $\theta_i'(E_{\Pi_i}(\overline{x}_i(a_i', a_{-i})) - E_{\Pi_i}(\overline{x}_i(a_i'', a_{-i}))) > E_{\Pi_i}(\overline{t}_i(a_i', a_{-i})) - E_{\Pi_i}(\overline{t}_i(a_i'', a_{-i}))$. Therefore, $\theta_i' E_{\Pi_i}(\overline{x}_i(a_i', a_{-i})) - E_{\Pi_i}(\overline{t}_i(a_i', a_{-i})) > \theta_i' E_{\Pi_i}(\overline{x}_i(a_i'', a_{-i})) - E_{\Pi_i}(\overline{t}_i(a_i'', a_{-i}))$. Therefore, $V_i(a_i', \theta_i'') > V_i(a_i'', \theta_i'')$ implies that $V_i(a_i', \theta_i') > V_i(a_i'', \theta_i')$. Similar arguments establish the result with weak inequalities. Therefore, the set of actions that maximize* ex interim *expected utility is increasing in strong set order as a function of the valuation, by Milgrom and Shannon (1994).*

Similarly to with dependent valuations, letting $a_i \in \mathcal{A}_i^d$ be given and considering any $a_i' \in \rho_{Li}(a_i)$, any valuation θ_i' consistent with using a_i' satisfies $\theta_i' \geq \Lambda_i(a_i')$. Moreover, since $a_i' \in \mathcal{A}_i^d$ by construction, there is indeed some valuation θ_i' that uses action a_i'. Now consider any $\tilde{\theta}_i < \Lambda_i(a_i')$. Since $\tilde{\theta}_i \not\geq \Lambda_i(a_i')$ by construction, valuation $\tilde{\theta}_i$ cannot use action a_i'. Suppose that, in the sense of Assumption 4, player i with valuation $\tilde{\theta}_i$ uses action $\tilde{a}_i$. Suppose, in order to prove a contradiction, that $\tilde{a}_i \geq a_i'$. Since $\tilde{\theta}_i < \Lambda_i(a_i') \leq \theta_i'$, because the set of actions that maximize ex interim *expected utility is increasing in strong set order, it must be that $\tilde{a}_i$ maximizes* ex interim *expected utility when the valuation is θ_i' and a_i' maximizes* ex interim *expected utility when the valuation is $\tilde{\theta}_i$. But, by the above, a_i' does not maximize* ex interim *expected utility when the valuation is $\tilde{\theta}_i$, so it must be that $\tilde{a}_i < a_i'$. Thus, player i with valuation $\tilde{\theta}_i$ must use an action strictly less than a_i'. By the contrapositive, any action weakly greater than a_i' must correspond to a valuation weakly greater than $\Lambda_i(a_i')$. Consequently, because $a_i' \leq a_i$ by construction, the valuation θ_i corresponding to the use of action a_i must be weakly greater than $\Lambda_i(a_i')$. Since the above holds for any $a_i' \in \rho_{Li}(a_i)$, the valuation θ_i corresponding to action a_i must be weakly greater than $\sup_{a_i' \in \rho_{Li}(a_i)} \Lambda_i(a_i')$. By similar arguments, the valuation θ_i corresponding to action a_i must be weakly less than $\inf_{a_i' \in \rho_{Ui}(a_i)} \Lambda_i(a_i')$. Because valuations are between Θ_{Li} and Θ_{Ui}, by Assumption 8, the valuation corresponding to action a_i is between $\omega_{Li}(a_i)$ and $\omega_{Ui}(a_i)$.* ★ □

Proof of Theorem 2. From Assumptions 9, 10, 11, and 12, let $\mathcal{E}_i = (\text{int}(\mathcal{A}_{i,\text{cont}}))^C \cup \mathcal{E}_{i,d} \cup \mathcal{E}_{i,r} \cup \mathcal{E}_{i,m}$ and $\mathcal{E} = \prod_i \mathcal{E}_i$. It follows that $P(A \in \mathcal{E}) = 0$. Then $P(\theta \in B) = P(\theta \in B, A \in \mathcal{E}^C) + P(\theta \in B, A \in \mathcal{E}) = P(\theta \in B, A \in \mathcal{E}^C) = P(\theta \in B | A \in \mathcal{E}^C)$ for any Borel set B, so it is enough to restrict the identification problem to recovering the distribution of θ from actions in $\mathcal{E}^C$. By Assumptions 3, 4, 9, and 10, Equation 2 is the necessary condition for any action used by player i in $\mathcal{A}_i^d \cap \text{int}(\mathcal{A}_{i,\text{cont}}) \cap \mathcal{E}_{i,d}^C$. See Footnote 18 for the arguments. By Assumptions 1, 6, and 9, conditioning on θ_i is equivalent to conditioning on $A_i = a_i(\theta_i)$, so by Assumption 5, Equation 5 is valid for actions in $\mathcal{A}_i^d \cap \text{int}(\mathcal{A}_{i,\text{cont}}) \cap \mathcal{E}_{i,d}^C$. Under Assumption 12, Equation 10 is valid for all actions used by player i in $\mathcal{A}_i^d \cap \text{int}(\mathcal{A}_{i,\text{cont}}) \cap \mathcal{E}_{i,d}^C \cap \mathcal{E}_{i,m}^C$. By Assumption 11, $\Psi_i(a_i)$ is point identified for all $a_i \in \mathcal{A}_i^d \cap \text{int}(\mathcal{A}_{i,\text{cont}}) \cap \mathcal{E}_{i,d}^C \cap \mathcal{E}_{i,m}^C \cap \mathcal{E}_{i,r}^C$. Therefore, the identification result obtains.

Independent valuations *Under Assumption 2, the following adjustments are made to the proof. Equation 2 need not condition on θ_i since beliefs are independent of valuation. Similarly, Equation 5 is valid without conditioning on A_i, so Assumption 6 need not be used.* ★ □

Proof of Theorem 3. The arguments are exactly the same as the arguments in the proof of Theorem 2, except restricted to identifying θ_i from A_i using $\Psi_i(\cdot)$. □

Proof of Lemma 2. By Assumption 5, $\theta_i E_{\Pi_i}(\overline{x}_i(a_i, a_{-i})|\theta_i') - E_{\Pi_i}(\overline{t}_i(a_i, a_{-i})|\theta_i') = \theta_i E_{\Pi_i}(\overline{x}_i(a_i, a_{-i}(\theta_{-i}))|\theta_i') - E_{\Pi_i}(\overline{t}_i(a_i, a_{-i}(\theta_{-i}))|\theta_i')$, because the distribution of $A_{-i}|\theta_i'$ is the same as the distribution of $a_{-i}(\theta_{-i})|\theta_i'$. Under Assumption 6, and the condition that $\theta_i \overline{x}_i(a_i^*, a_{-i}) - \overline{t}_i(a_i^*, a_{-i})$ is a weakly decreasing function of a_{-i} for a_i^* as in the statement of the lemma, $\theta_i \overline{x}_i(a_i^*, a_{-i}(\theta_{-i})) - \overline{t}_i(a_i^*, a_{-i}(\theta_{-i}))$ is a weakly decreasing function of θ_{-i}. Under affiliation, by standard properties of affiliated random variables (e.g., Milgrom (2004, Theorem 5.4.5)), it follows that $\theta_i E_{\Pi_i}(\overline{x}_i(a_i^*, a_{-i})|\theta_i') - E_{\Pi_i}(\overline{t}_i(a_i^*, a_{-i})|\theta_i')$ is a weakly decreasing function of θ_i'. Alternatively, under monotonicity of $\theta_{-i}|\theta_i$ in the usual multivariate stochastic order, by standard properties of the usual multivariate stochastic order (e.g., Shaked and Shanthikumar (2007, Chapter 6)), it follows that $\theta_i E_{\Pi_i}(\overline{x}_i(a_i^*, a_{-i})|\theta_i') - E_{\Pi_i}(\overline{t}_i(a_i^*, a_{-i})|\theta_i') \geq \theta_i E_{\Pi_i}(\overline{x}_i(a_i^*, a_{-i})|\theta_i'') - E_{\Pi_i}(\overline{t}_i(a_i^*, a_{-i})|\theta_i'')$ for $\theta_i' \leq \theta_i''$. □

Proof of Lemma 3. By definition, $\overline{x}_i(a) = E(\tilde{x}_i(a)) = E(\tilde{x}_i(a)|A_i = a_i, A_{-i} = a_{-i}) = E(X_i|A_i = a_i, A_{-i} = a_{-i})$ and $\overline{t}_i(a) = E(\tilde{t}_i(a)) = E(\tilde{t}_i(a)|A_i = a_i, A_{-i} = a_{-i}) = E(T_i|A_i = a_i, A_{-i} = a_{-i})$. Under the conditions of the lemma, for $a = (a_1, a_2, \ldots, a_N)$ such that $a_j \in \mathcal{A}_j^d$ for all j, it holds that also $a \in \mathcal{A}^d$ and therefore $\overline{x}_i(a)$ and $\overline{t}_i(a)$ are point identified by the previous expressions in terms of conditional expectations, conditional on a discrete variable. Then, consider $E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i')$ and suppose that $a_i \in \mathcal{A}_i^d$ and $z_i' \in \mathcal{A}_i^d$. Obviously, the support of $A_{-i}|(A_i = z_i')$ is a subset of the support of A_{-i}, and $a_i \in \mathcal{A}_i^d$ by assumption, and therefore $\overline{x}_i(a_i, a_{-i})$ is point identified at all points relevant to $E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i')$. And of course the distribution of $A_{-i}|(A_i = z_i')$ is identified since $z_i' \in \mathcal{A}_i^d$. Therefore, $E_P(\overline{x}_i(a_i, A_{-i})|A_i = z_i')$ is point identified. It is similar for $E_P(\overline{x}_i(z_i, A_{-i})|A_i = z_i'')$, $E_P(\overline{t}_i(a_i, A_{-i})|A_i = z_i')$, and $E_P(\overline{t}_i(z_i, A_{-i})|A_i = z_i'')$. Therefore, there is game-structure identification per Definition 5. □

Proof of Theorem 4. By Assumption 4, Equation 21 is a necessary condition for any action $\tilde{a}_i(\theta_i)$ used by player i. Then, under Assumption 5, Equation 22 is an equivalent necessary condition. Then, under Assumptions 6, 13, and 14, Equation 26 is valid. Under Assumption 6, given that $z_i' < a_i(\theta_i) < z_i''$ are all used in the data, all elements of $\Theta_i(z_i')$ are less than all elements of $\Theta_i(a_i(\theta_i))$, and all elements of $\Theta_i(a_i(\theta_i))$ are less than all elements of $\Theta_i(z_i'')$, where $\Theta_i(\cdot)$ is defined in Equation 23. In particular, $\theta_i \in \Theta_i(a_i(\theta_i))$, all elements of $\Theta_i(z_i')$ are less than θ_i, and θ_i is less than all elements of $\Theta_i(z_i'')$. Then, combining Equations 24 and 25 with Equation 26, Equation 27 is valid. Equations 28, 29, 30, and 31 follow immediately, using Assumption 8. Then, by Assumption 6 and arguments similar to those used in the proof of Theorem 1, the valuation corresponding to a_i must be between $\sup_{a_i' \leq a_i, a_i' \in \mathcal{A}_i^d} \Phi_{Li}(a_i')$ and $\inf_{a_i' \geq a_i, a_i' \in \mathcal{A}_i^d} \Phi_{Ui}(a_i')$. Because the identification result from Theorem 1 also holds under these conditions, the valuation corresponding to a_i must be between $\Upsilon_{Li}(a_i)$ and $\Upsilon_{Ui}(a_i)$ defined in Equation 34. Therefore, considering the joint distribution of $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$ and corresponding observed actions $A = (A_1, A_2, \ldots, A_N)$, it holds for all realizations that, for each $i \in \{1, 2, \ldots, N\}$, $\Upsilon_{Li}(A_i) \leq \theta_i \leq \Upsilon_{Ui}(A_i)$. Consequently, the partial identification result in the usual multivariate stochastic order follows from Shaked and Shanthikumar (2007, Theorem 6.B.1).

Independent valuations *Under Assumption 2, the following adjustments are made to the proof. Under Assumption 2, Equation 21 need not condition on θ_i since beliefs do not depend on valuation. Similarly, Equation 22 need not condition on θ_i. Thus, Equations 32 and 33 are valid bounds for the valuation, even without Assumptions 6, 13, and 14. Then, by Assumption 7 and arguments similar to those used in the proof of Theorem 1 under Assumption 2, the valuation corresponding to a_i must be between $\sup_{a_i' \leq a_i, a_i' \in \mathcal{A}_i^d} \Xi_{Li}(a_i')$ and $\inf_{a_i' \geq a_i, a_i' \in \mathcal{A}_i^d} \Xi_{Ui}(a_i')$. Because the identification result from Theorem 1 also holds under these conditions, the valuation corresponding to a_i must be between $\Gamma_{Li}(a_i)$ and $\Gamma_{Ui}(a_i)$ defined in Equation 35.* ★ □

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